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Orbital diameters of the symmetric and alternating groups

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Abstract For a primitive group G acting on a finite set Ω, we define the orbital diameter to be the maximum of the diameters of all orbital graphs of G. In this paper, we study the orbital diameters of symmetric and alternating groups. We give necessary numerical conditions for the orbital diameter to be bounded by some constant c and give precise descriptions of the actions for which the orbital diameter is bounded by 5. For each primitive action, we also either determine all orbital graphs of diameter 2 or give descriptions of infinite families of orbital graphs of diameter 2.

Keywords Orbital graph · Primitive · Symmetric · Alternating · Diameter

1 Introduction

Let G be a finite transitive permutation group acting on a set Ω. We will denote such a group by (G, Ω). The *orbitals* of (G, Ω) (or just G) are the orbits of the natural action of G on $\Omega \times \Omega$. The *diagonal* orbital is $\Delta_0 := \{(\alpha, \alpha) : \alpha \in \Omega\}$. To each orbital Δ, we associate the *paired* orbital $\Delta^* := \{(\beta, \alpha) : (\alpha, \beta) \in \Delta)\}$; Δ is called *self-paired* if $\Delta = \Delta^*$. For a given orbital Δ, the *orbital graph* $\Gamma_{\Delta\cup\Delta^*}$ is defined to be the undirected graph whose vertex set is Ω and edge set is $\Delta \cup \Delta^*$. Henceforth we will assume that $\Delta \neq \Delta_0$ and write Γ_Δ if Δ is self-paired. Note that orbital graphs belong to a wider family of graphs known as edge-transitive graphs. By a theorem of D. G. Higman, a permutation group (G, Ω) is primitive if and only if all the orbital graphs are connected [see [5], (1.12)]. For a graph Γ, we will use $V(\Gamma)$ to denote the vertex set of Γ. The *valency* of $\alpha \in V(\Gamma)$ is the number of edges that are connected to

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α; if all vertices of Γ have the same valency v, then v is defined to be the valency of Γ. For a connected graph Γ, the *distance* between two vertices u and v is the length of a shortest path between u and v and will be denoted by $d_\Gamma(u, v)$; we will always assume that $u \neq v$ and use the notation $d(u, v)$ if it is clear which graph is in question. The *diameter* of a connected graph is the greatest distance between any two vertices and will be denoted by $\mathrm{diam}(\Gamma)$. The *orbital diameter* of a primitive permutation group (G, Ω) is the maximum of the diameters of the orbital graphs of (G, Ω) and will be denoted by $\mathrm{diam}_\mathrm{O}(G, \Omega)$. If $\mathcal{C}$ is a class of finite primitive permutation groups, then $\mathcal{C}$ is said to be *bounded* if there exists a positive integer d such that $\mathrm{diam}_\mathrm{O}(G, \Omega) \leq d$ for all (G, Ω) in $\mathcal{C}$. In this paper, we study the orbital diameters of finite symmetric and alternating groups in their primitive actions. The study is motivated by a paper [7] by Liebeck, Macpherson and Tent in which the authors describe infinite bounded classes of finite primitive permutation groups. Let $\mathcal{C}$ be such an infinite class. The main theorem of [7] gives tight structural information about the primitive groups in $\mathcal{C}$ depending on the type of the primitive group as classified in the O'Nan–Scott theorem [8]. In particular, if k is a fixed integer and $\mathcal{C}$ is an infinite class of finite symmetric groups S_n, or alternating groups A_n, with natural action on $I^{\{k\}}$, the set of k-subsets of $I := \{1, \ldots, n\}$, then $\mathcal{C}$ is bounded. Moreover, this is the only primitive action of S_n and A_n which gives rise to a bounded infinite class.

In this paper, we present versions of the results in [7] for finite symmetric and alternating groups with precise numerical bounds on the orbital diameters in question (see Theorem 1.1). As a consequence, we classify the primitive actions of the symmetric and alternating groups for which the orbital diameter is at most 5 (see Theorem 1.2). In addition, we prove results classifying orbital graphs of diameter 2 (see Theorem 1.3).

We would like to mention here that the results in this paper are part of a more general programme of classifying bounded infinite families of finite primitive permutation groups. Such results of the author on primitive actions of simple groups of Lie type will be forthcoming.

We will use the notation $(G : H)$ to denote the set of right cosets of a subgroup H in a group G.

Let $G := S_n$ or A_n and $I := \{1, \ldots, n\}$. Henceforth we will assume that $n \geq 5$. If H is a maximal subgroup of G then one of the following holds:

1. H is intransitive on I, and hence $H \cong (S_k \times S_{n-k}) \cap G$, where $1 \leq k < n/2$.
2. H is imprimitive on I, and hence $H \cong (S_k \wr S_l) \cap G$, where $n = kl$ and $k > 1$, $l > 1$.
3. H is primitive on I.

Let $I = \{1, \ldots, n\}$. If H is intransitive on I, then the action of $G := S_n$ or A_n on the set $(G : H)$ is equivalent to the action of G on the set $I^{\{k\}}$, of k-subsets of I. The orbitals of $(G, I^{\{k\}})$ are $\Delta_i^n := \{(A, B) : |A \cap B| = i\}$, for $i \in \{0, \ldots, k-1\}$. The associated orbital graphs are $\Gamma_0^n, \ldots, \Gamma_{k-1}^n$, where for $i \in \{0, \ldots, k-1\}$, the graph Γ_i^n has vertex set $I^{\{k\}}$ and edge set Δ_i^n. Note that for $k = 1$, there is only one orbital graph Γ_0^n which has diameter 1; henceforth, we will assume that $k > 1$. Lastly, if $n = 2k$, then $H := S_k \times S_k$ is not a maximal subgroup of S_n; however, the orbital graphs of the action of S_n on the set $(S_n : H)$ are still Γ_i^n, for $i \in \{0, \ldots, k-1\}$, and Γ_i^n is connected for all $i \neq 0$.

If H is imprimitive on I, then the action of $G := S_n$ or A_n on $(G : H)$ is equivalent to the action of G on the set $I^{(k,l)}$, of (k,l)-partitions of I; here a (k,l)-partition of I is a partition of I into l sets $A_1, \ldots, A_l$, each of size k. Henceforth such a (k,l)-partition of I will be denoted by $A_1|\ldots|A_l$. Let $A := A_1|\ldots|A_l$ and $B := B_1|\ldots|B_l$ be two (k,l)-partitions of I. We will denote by I_{AB} the $l \times l$ matrix with entries $(I_{AB})_{ij} := |A_i \cap B_j|$. Note that the row and columns sums of I_{AB} are k, and for every $l \times l$ matrix M whose entries are non-negative integers, and whose row and column sums are k, there exists a pair (A, B) of (k,l)-partitions of I such that $I_{AB} = M$. Let $A, B, C, D \in I^{(k,l)}$. Then the pairs (A, B) and (C, D) lie in the same orbital of S_n if and only if there exist $l \times l$ permutation matrices P and Q such that $I_{CD} = P\, I_{AB}\, Q$, i.e. I_{AB} and I_{CD} are equal up to permutation of rows and columns (see Proposition 4.1). Let $\mathcal{N}$ denote the set of all $l \times l$ matrices with non-negative integer entries, and row and column sums equal to k. Define an equivalence relation $\sim$ on $\mathcal{N}$, where $M \sim M'$ if and only if M' can be obtained from M by a permutation of rows and columns. The orbitals of $(G, I^{(k,l)})$ can then be described as

$$\Delta_{[M]} := \left\{(A, B) : A, B \in I^{(k,l)}, I_{AB} \in [M]\right\}$$

where $M \in \mathcal{N}$, and $[M]$ is the equivalence class of M under the equivalence relation $\sim$. Notice that for an orbital $\Delta_{[M]}$, the paired orbital $\Delta^*_{[M]}$ is $\Delta_{[M^{\mathrm{T}}]}$, where T in the superscript denotes taking the transpose of the matrix. The corresponding orbital graphs in this case are $\Gamma_{[M]\cup[M^{\mathrm{T}}]}:=\Gamma_{\Delta_{[M]}\cup\Delta^*_{[M]}}$. If $M^{\mathrm{T}} = M$, then we will denote the corresponding orbital graph by $\Gamma_{[M]}$.

In the case where H is primitive on I and $G := S_n$, we will assume that $H \neq A_n$, since $\mathrm{diam_O}(G, (G : A_n)) = 1$, for any n.

We now state our main results.

Theorem 1.1 *Let (G, Ω) be a finite primitive permutation group where $G := S_n$ or A_n, and let $I = \{1, \ldots, n\}$. Suppose there exists an integer $c \geq 1$ such that* $\mathrm{diam_O}(G, \Omega) \leq c$.

1. *If $\Omega = I^{\{k\}}$, where $1 \leq k < n/2$, then $k \leq c$.*
2. *If $\Omega = I^{(k,l)}$, where $n = kl$ with $l > 1$ and $k > 1$, then $l \leq 2c$ and $k \leq 2c + 1$, and in particular, $n \leq 2c(2c + 1)$.*
3. *If $\Omega = (G : H)$, where H is primitive on I, then $n < 2^{\epsilon(c+1)+3}$ with $\epsilon := 1.04$.*

Notice that Theorem 1.1 agrees with the results in [7], i.e. part 1 is the only case for which n is unbounded. The proofs of Theorem 1.1(1), (2) and (3) can be found on Pages 9, 18, and 26, respectively.

For particular values of the bound c, it is possible to be more specific about the precise actions that occur in Theorem 1.1(2) and (3). Here is such a result where $c = 5$.

Theorem 1.2 *Let (G, Ω) be a finite primitive permutation group where $G := S_n$ or A_n, and let $I = \{1, \ldots, n\}$.*

1. *If $\Omega = I^{\{k\}}$, where $1 \leq k < n/2$, then* $\mathrm{diam_O}(G, \Omega) \leq 5$ *if and only if $k \leq 5$.*
2. *If $\Omega = I^{(k,l)}$, where $n = kl$ with $l > 1$ and $k > 1$, then* $\mathrm{diam_O}(G, \Omega) \leq 5$ *if and only if the pair (k, l) is given by Table 1.*

Table 1 Pairs (k, l) with $\mathrm{diam}_{\mathrm{O}}(G, I^{(k,l)}) \leq 5$

l	2	3	4	5	6
k	≤ 11	≤ 5	≤ 3	2	2

Table 2 Pairs (n, H) with $\mathrm{diam}_{\mathrm{O}}(S_n, \Omega) \leq 5$, where $\Omega := (S_n : H)$ and H is primitive

n	H	Rank	Diameters of orbital graphs	Valencies of orbital graphs
5	$AGL_1(5)$	2	1	5
6	$PGL_2(5)$	2	1	5
7	$AGL_1(7)$	7	3, 3, 3, 2, 2, 2	7, 14, 14, 21, 21, 42
8	$PGL_2(7)$	5	3, 2, 2, 2	14, 21, 28, 56
9	$AGL_2(3)$	9	5, 3, 3, 3, 3, 2,	8, 27, 36, 48, 144,
			2, 2	216, 216
10	$P\Gamma L_2(9)$	10	4, 4, 3, 3, 3, 2	20, 45, 90, 144, 180, 240
			2, 2, 2	360, 720, 720
14	$PGL_2(13)$	19,299	?	?
18	$PGL_2(17)$	?	?	?

Table 3 Pairs (n, H) with $\mathrm{diam}_{\mathrm{O}}(A_n, \Omega) \leq 5$, where $\Omega := (A_n : H)$ and H is primitive

n	H	Rank	Diameters of orbital graphs	Valencies of orbital graphs
5	D_{10}	2	1	5
6	$PSL_2(5)$	2	1	5
7	$PSL_2(7)$	2	1	14
8	$2^3 : PSL_2(7)$	2	1	14
9	$3^2 : 2A_4$	12	5, 3, 3, 3, 3, 2	8, 48, 27, 36, 72, 144,
			3, 2, 2	72, 216, 216
9	$P\Gamma L_2(8)$	3	2, 2	56, 63
10	M_{10}	12	4, 4, 3, 3, 3, 3	20, 45, 90, 90, 90, 144,
			2, 2, 2, 2	240, 360, 720, 720
11	M_{11}	5	2, 2, 2, 2	110, 330, 495, 1584
12	M_{12}	4	2, 2, 2	440, 495, 1584
14	$PSL_2(13)$	?	?	?
24	M_{24}	?	?	?

3. *If* $\Omega = (G : H)$*, where* H *is primitive on* I*, then* $\mathrm{diam}_{\mathrm{O}}(G, \Omega) \leq 5$ *if and only if the pair* (n, H) *is given by Tables 2 and 3; where the 'if' direction is only known to hold for values of* $n \leq 12$ *(see the remark below).*

Remark Tables 2 and 3 include:

1. the rank of the action, the diameter of the orbital graphs and their valencies in each case,
2. four cases with $n = 14$, 18 or 24, marked with question marks, where we have been unable to compute the orbital diameters.

Table 4 Pairs (n, H) such that (S_n, Ω) has an orbital graph of diameter 2, where $\Omega := (S_n : H)$ and H is primitive

n	H	Rank	$\lvert\mathcal{D}_2\rvert$	Valencies of orbital graphs of diameter 2
7	$AGL_1(7)$	7	3	21, 21, 42
8	$PGL_2(7)$	5	3	21, 28, 56
9	$AGL_2(3)$	9	3	144, 216, 216
10	$P\Gamma L_2(9)$	10	4	240, 360, 720, 720

The proofs of Theorem 1.2(1), (2) and (3) can be found on Pages 9, 25, and 29, respectively.

We also analyse orbital graphs of diameter 2. For the case where $\Omega \neq I^{(k,l)}$, we find all orbital graphs of diameter 2. For $\Omega = I^{(k,l)}$, we find an example of an infinite family of orbital graphs of diameter 2; there could be more infinite families of orbital graphs of diameter 2, but classifying all of them could take significantly more effort.

Theorem 1.3 *Let (G, Ω) be a finite primitive permutation group where $G := S_n$ or A_n, and let $I = \{1, \ldots, n\}$.*

1. *If $\Omega = I^{\{k\}}$, where $1 < k < n/2$, then $\operatorname{diam}(\Gamma_i^n) = 2$ if and only if one of the following holds:*
   - *$i = 0$ and $n \geq 3k - 1$, or*
   - *$i \in \{1, \ldots, \lfloor k/2 \rfloor\}$ and $n \geq 3k - 2i$.*
2. *If $\Omega = I^{(k,k)}$ with $n = k^2$ and $k \geq 3$, let $\Gamma_{[M]}$ be the orbital graph where M is the $k \times k$ matrix with all entries equal to 1. Then $\operatorname{diam}(\Gamma_{[M]}) = 2$.*
3. *If $\Omega = (G : H)$, where H is primitive on I, then Tables 4 and 5 specify the precise pairs (n, H) for which (G, Ω) has an orbital graph of diameter 2 (the case n = 24 is still just a possibility; see the remark below).*

Remark

1. Tables 4 and 5 include the rank of the action, the size of the set $\mathcal{D}_2$, of orbital graphs of diameter 2 and the valencies of the orbital graphs in each case.
2. Table 5 contains the case $n = 24$, marked with a question mark, where we have been unable to determine or disprove the existence of an orbital graph of diameter 2.
3. The notation $x^{[y]}$ has been used in the 'valencies' column of Table 5 to denote that the valency x appears y times.

The proofs of Theorem 1.3(1), (2) and (3) can be found on Pages 9, 25, and 29, respectively.

The proofs of the main results require some analysis of the three types of primitive actions of $G := S_n$ or A_n. In the first case, namely the action of G on Ω where $\Omega := I^{\{k\}}$, some work has been done in [2] on finding the diameters of the corresponding orbital graphs. However, we believe that the main result in [2] has an error and state the correct result below.

Table 5 Pairs (n, H) such that (A_n, Ω) has an orbital graph of diameter 2, where $\Omega := (A_n : H)$ and H is primitive

n	H	Rank	$\lvert\mathcal{D}_2\rvert$	Valencies of orbital graphs of diameter 2
9	$3^2 : 2A_4$	12	3	144, 216, 216
9	$P\Gamma L_2(8)$	3	2	56, 63
10	M_{10}	12	4	240, 360, 720, 720
11	M_{11}	5	4	110, 330, 495, 1584
12	M_{12}	4	3	440, 495, 1584
13	$PSL_3(3)$	126	28	$5616^{[3]}$, $11{,}232^{[25]}$
15	$PSL_4(2)$	1687	≥ 8	$40{,}320^{[8]}$, ...
16	$2^3 : PSL_2(7)$	151	79	$40{,}320^{[1]}$, $80{,}640^{[2]}$, $53{,}760^{[1]}$, $64{,}512^{[1]}$, $161{,}280^{[7]}$, $107{,}520^{[3]}$, $322{,}560^{[39]}$, $645{,}120^{[25]}$
24	M_{24}	?	?	?

Theorem 1.4 *Let $k > 1$ and $n \geq 2k + 1$ be integers. For $i \in \{0, \ldots, k-1\}$, let Γ_i^n be as defined previously. Then the following hold:*

1. *For $i = 0$, $\mathrm{diam}(\Gamma_0^n) = \lceil \frac{n-k-1}{n-2k} \rceil$.*
2. *For $i \in \{1, \ldots, \lfloor k/2 \rfloor\}$,*
   (a) $\mathrm{diam}(\Gamma_i^n) = 2 \Leftrightarrow n \geq 3k - 2i$,
   (b) $\mathrm{diam}(\Gamma_i^n) = 3 \Leftrightarrow \frac{5(k-i)-1}{2} \leq n < 3k - 2i$,
   (c) $\mathrm{diam}(\Gamma_i^n) = \lceil \frac{n-k+i-1}{n-2k+2i} \rceil > 3 \Leftrightarrow n < \frac{5(k-i)-1}{2}$.
3. *For $i > \lfloor k/2 \rfloor$, $\mathrm{diam}(\Gamma_i^n) = \lceil \frac{k}{k-i} \rceil$.*

The error occurs in line 2 on p. 6648 of [2]. According to [2], for any $i > \lfloor k/2 \rfloor$, $\mathrm{diam}(\Gamma_i^n) = 3$, and in Theorem 7, a construction for a path of length 3 is specified for any two vertices A and B with $m := |A \cap B| < 2i - k$. This involves constructing a set $Z = Z(n, k, i, m)$ which is, however, impossible to construct for many values of n, k, i and m; for example, let $(n, k, i, m) := (2k + 1, k, k - 1, 0)$, where $k \geq 4$, then $|Z| := 2k - 3i + m = 3 - k \leq -1$. In fact $\mathrm{diam}(\Gamma_{k-1}^n) = k$.

Note that in order to prove Theorem 1.3(1), we do not require Theorem 1.4 in its entirety; however, we give a proof of the whole theorem for completeness. The proof of Theorem 1.4(1) and (2c) can be found on Page 14, and the proofs of (2a), (2b) and (3) can be found on Pages 9, 12, and 16, respectively.

To study the orbital graphs for the action of G on Ω, where $\Omega := I^{(k,l)}$ and $n = kl$, we will need Theorem 1.4(2) as well as the following results.

Theorem 1.5 *Let $G := S_n$ or A_n and $\Omega := I^{(k,l)}$, where $n = kl$ with $k > 1$ and $l > 1$. Then $diam_{\mathrm{O}}(G, \Omega) \geq \frac{l}{2} \lfloor \frac{k}{2} \rfloor$.*

The proof of Theorem 1.5 can be found on Page 18.

Theorem 1.6 *Let* $G := S_n$ *or* A_n *and* $\Omega := I^{(k,2)}$, *where* $n = 2k$ *with* $k > 1$. *For* $i \in \{1, \dots, \lfloor k/2 \rfloor\}$, *let*

$$M_i := \begin{pmatrix} i & k-i \\ k-i & i \end{pmatrix}.$$

Then the following hold:

1. *For* $k = 2, 3$, $\mathrm{diam}(\Gamma_{[M_1]}) = 1$.
2. *For* $k \geq 4$,
   (a) $\mathrm{diam}(\Gamma_{[M_i]}) = 2$ *if and only if* $\frac{k-1}{4} \leq i \leq \frac{k}{2}$, *and*
   (b) *if* $\mathrm{diam}(\Gamma_{[M_i]}) \neq 2$, *then* $\mathrm{diam}(\Gamma_{[M_i]}) = \left\lceil \frac{k-1}{2i} \right\rceil$.

The proof of Theorem 1.6(1) and (2a) can be found on Page 22, and the proof of (2b) can be found on Page 24.

2 Computing diameters using MAGMA

Some of the results in this paper rely on computations which have been done using MAGMA [1]. In this section, we describe the methods used for the computations and their implementation in MAGMA.

2.1 Method CAM: using collapsed adjacency matrices

Let G be a transitive permutation group acting on a finite set Ω. Let $\Omega_1 := \{\alpha\}, \Omega_2, \dots, \Omega_r$ be the distinct orbits of G_α on Ω, i.e. the *suborbits* of G, with representatives $\alpha_1 := \alpha, \alpha_2, \dots, \alpha_r$, where r is the *rank* of the action. The suborbits of G are in one-to-one correspondence with the orbitals of G; a suborbit β^{G_α} corresponds to the orbital $(\alpha, \beta)^G := \{(\alpha, \beta)^g : g \in G\}$, where $(\alpha, \beta)^g$ denotes the action of g on (α, β). Let E be a union of orbitals of (G, Ω). Define an $r \times r$ matrix A_E where for $k, j \in \{1, \dots, r\}$,

$$(A_E)_{kj} := |E(\alpha_k) \cap \Omega_j|$$

and $E(\alpha_k) := \{\gamma \in \Omega : (\alpha_k, \gamma) \in E\}$. Note that A_E does not depend on the choice of the suborbit representative α_k. If $E = \Delta \cup \Delta^*$ where Δ is an orbital of G, then A_E is called the *collapsed adjacency matrix* for the orbital graph $\Gamma_{\Delta\cup\Delta^*}$ (with respect to the ordering of the suborbits). The corresponding *collapsed graph* is defined to be the undirected graph $\Gamma^C_{\Delta\cup\Delta^*}$, with vertex set $\{1, \dots, r\}$, and in which (k, j) is an edge if $(A_{\Delta\cup\Delta^*})_{kj} > 0$. When computing the diameter of an orbital graph, it is advantageous to work with the corresponding collapsed graph since it has significantly fewer vertices than the orbital graph but in fact the same diameter. The following lemma, which is based on Theorem 3.2 in [10], justifies the switch from $\Gamma_{\Delta\cup\Delta^*}$ to $\Gamma^C_{\Delta\cup\Delta^*}$.

Lemma 2.1 *Let* (G, Ω) *be a finite primitive permutation group and* Γ *an orbital graph of* G. *Then* $\mathrm{diam}(\Gamma) = \mathrm{diam}(\Gamma^C)$.

The following result aids the computation of collapsed adjacency matrices; see Proposition 2.2 in [10].

Lemma 2.2 *Let* (G, Ω) *be a finite primitive permutation group and* Δ *be an orbital of* G *which is not self-paired. Then* $A_{\Delta \cup \Delta^*} = A_\Delta + A_{\Delta^*}$.

2.2 Implementation of CAM in MAGMA

MAGMA does not have an in-built function to calculate collapsed adjacency matrices. We can, however, use an in-built function to define the group G and iterate over the maximal subgroups of G to find the required subgroup $H \cong G_\alpha$. We can then use more in-built functions to firstly construct the permutation group (G, Ω) where $\Omega := (G : H)$, and secondly, find the suborbits of G, with representatives $\alpha_1 := \alpha, \alpha_2, \ldots, \alpha_r$. Let $\Delta := (\alpha, \beta)^G$ be an orbital of G. We calculate A_Δ by implementing the method described in [10, p. 24]. If Δ is self-paired, then A_Δ is the collapsed adjacency matrix for the orbital graph $\Gamma_{\Delta \cup \Delta^*}$. If Δ is not self-paired, then we can use the function *trace* : $\Omega \to G$ (constructed whilst implementing the method described in [10]) where *trace*$(\gamma) = g$ means that $\alpha^g = \gamma$, to find $g \in G$ such that $\beta^g = \alpha$. Then $\Delta^* = (\alpha, \alpha^g)^G$, and we can again use the method in [10] to compute the matrix A_{Δ^*}. The collapsed adjacency matrix for the orbital graph $\Gamma_{\Delta \cup \Delta^*}$ is then $A_\Delta + A_{\Delta^*}$ by Lemma 2.2. Once we have a collapsed adjacency matrix A_E where $E := \Delta \cup \Delta^*$, the diameter of Γ_E^C (and hence the diameter of Γ_E by Lemma 2.1) is the least number of times A_E has to be multiplied by itself before each non-diagonal entry has taken a value greater than zero at least once.

The implemented method works well when $|G : H|$ is small. For large values of $|G : H|$, the method fails at the very first step of constructing the permutation group (G, Ω).

3 Action of S_n on k-subsets of I

In this section, we prove Theorems 1.1(1), 1.2(1), and 1.3(1). We then prove Theorem 1.4 which is used to deduce Theorem 1.6 in Sect. 4.

Proposition 3.1 *Let* (G, Ω) *be a finite primitive permutation group where* $G := S_n$ *or* A_n, $I = \{1, \ldots, n\}$, *and* $\Omega := I^{\{k\}}$ *with* $1 \leq k < n/2$. *Then* $\mathrm{diam}_\mathrm{O}(G, \Omega) \leq c$ *if and only if* $k \leq c$.

Proof Suppose $\mathrm{diam}_\mathrm{O}(G, \Omega) \leq c$. Recall that in the orbital graph Γ^n_{k-1}, two k-subsets are joined by an edge if they intersect in $k - 1$ points. Observe that this graph has diameter k, and so $k \leq c$. Conversely, suppose $k \leq c$ and let Γ be an orbital graph of (G, Ω). For a k-subset A and an integer $j \geq 1$ define

$$\Delta_j(A) := \{B : d_\Gamma(A, B) = j\}.$$

If $B \in \Delta_j(A)$ and $g \in G_A$, then $B^g \in \Delta_j(A)$, hence $B^{G_A} \subseteq \Delta_j(A)$. Therefore $\Delta_j(A)$ is a union of non-trivial orbits of G_A on Ω. There are k non-trivial orbits of G_A on Ω, and therefore $\mathrm{diam}(\Gamma) \leq k \leq c$. □

Proof of Theorem 1.1(1) and Theorem 1.2(1) Theorem 1.1(1) follows from Proposition 3.1, and Theorem 1.2(1) is a special case of Theorem 1.1(1). □

We now prove Theorem 1.3(1). Recall that for $i \in \{0, \dots, k-1\}$, the graph Γ_i^n has vertex set $I^{\{k\}}$ where $I = \{1, \dots, n\}$, and edge set $\{(A, B) : |A \cap B| = i\}$. Henceforth, for two sets A and B, the notation $A \backslash B$ will be used to refer to the complement of B in A. The following lemma follows from Lemmas 1 and 2 in [3] and the fact that Γ_i^n is a subgraph of the graph $K(n, k, i)$ from [3].

Lemma 3.2 *Let $n \geq 2k, i \in \{0, \dots, k-1\}$ ($i \neq 0$, if $n = 2k$), and $r \geq 1$. Let $A, B \in V(\Gamma_i^n)$.*

1. *If A and B are connected by a path of length $2r$, then*

$$|A \cap B| \geq (2r+1)k - rn - 2ri.$$

2. *If A and B are connected by a path of length $2r+1$, then*

$$|A \cap B| \leq rn - 2rk + (2r+1)i.$$

Lemma 3.3 *Let $n \geq 2k, i \in \{0, \dots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$), and $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$.*

1. *If $n < 3k - 2i$, then*
$$d(A, B) = 2 \Leftrightarrow m \geq 3k - n - 2i.$$
2. *If $n \geq 3k - 2i$, then $d(A, B) = 2$.*

Proof Let $I = \{1, \dots, n\}$. Suppose $n < 3k - 2i$. If $d(A, B) = 2$, then by Lemma 3.2(1), we have $m \geq 3k - n - 2i$. Conversely, assume $m \geq 3k - n - 2i$. If $i \leq k - m$, then choose a k-subset C consisting of i elements from $A \setminus B$, i elements from $B \setminus A$, and $k - 2i$ elements from $I \setminus (A \cup B)$ (note that $|I \setminus (A \cup B)| \geq k - 2i$, since $n \geq 3k - m - 2i$). If $i > k - m$, then choose a k-subset C consisting of $i - k + m$ elements from $A \cap B$, $k - m$ elements from $A \setminus B$, $k - m$ elements from $B \setminus A$, and $m - i$ elements from $I \setminus (A \cup B)$ (note that $|I \setminus (A \cup B)| \geq m - i$ since $n \geq 2k - i$). Then $A - C - B$ is a path of length 2, and hence $d(A, B) = 2$. This proves the first part of the lemma. Now suppose $n \geq 3k - 2i$. Then $m \geq 3k - n - 2i$, for all $m \geq 0$. Now as in the proof of the converse of the first part, we can choose a k-subset C such that $A - C - B$ is a path of length 2, and hence $d(A, B) = 2$. □

Corollary 3.4 *Let $n \geq 2k$ and $i \in \{0, \dots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$). Suppose that* $\mathrm{diam}(\Gamma_i^n) \geq 3$. *Then $n < 3k - 2i$ and for $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$,*
$$d(A, B) > 2 \Leftrightarrow m < 3k - n - 2i.$$

Proof of Theorem 1.3(1) and Theorem 1.4(2a) Note that $\mathrm{diam}(\Gamma_i^n) = 2$ if and only if for all $A \neq B \in V(\Gamma_i^n)$ such that $|A \cap B| \neq i$, it holds that $d(A, B) = 2$. Hence using Lemma 3.3, $\mathrm{diam}(\Gamma_0^n) = 2$ if and only if either $n \geq 3k$ or $3k - n = 1$, and if $i \in \{1, \dots, \lfloor k/2 \rfloor\}$, then $\mathrm{diam}(\Gamma_i^n) = 2$ if and only if $n \geq 3k - 2i$. Now suppose

Table 6 $v = 2r, r \geq 2$

	n	
	$< \frac{(2v-1)(k-i)}{v-1}$	$\geq \frac{(2v-1)(k-i)}{v-1}$
f	$(2r+1)k - rn - 2ri$	$(r-1)n - (2r-2)k + (2r-1)i + 1$
g	$(2r-1)k - (r-1)n - (2r-2)i - 1$	

Table 7 $v = 2r+1, r \geq 1$

	n	
	$< \frac{(2v-1)(k-i)}{v-1}$	$\geq \frac{(2v-1)(k-i)}{v-1}$
f	$0, r = 1$	
	$(r-1)n - (2r-2)k + (2r-1)i + 1, r \geq 2$	
g	$rn - 2rk + (2r+1)i$	$(2r+1)k - rn - 2ri - 1$

$i > \lfloor k/2 \rfloor$ and let $A, B \in V(\Gamma_i^n)$ with $|A \cap B| = 0$. If C is a common neighbour of A and B, then $|C| \geq 2i > k$ which contradicts the fact that $|C| = k$. Hence $d(A, B) > 2$ and $\mathrm{diam}(\Gamma_i^n) \geq 3$. □

We now continue proving the rest of Theorem 1.4. We will need to prove the following lemma.

Lemma 3.5 *Let $n \geq 2k$, $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$), and $d := \mathrm{diam}(\Gamma_i^n)$. Assume $d \geq 3$. Let $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$. Then for $v \in \{3, \ldots, d\}$,*

$$P(v): \quad d(A, B) = v \Leftrightarrow f(n, k, i, v) \leq m \leq g(n, k, i, v),$$

where f and g are as in Tables 6 and 7, for v even and v odd, respectively.

In order to prove Lemma 3.5, we now prove the following sequence of results.

Lemma 3.6 *Let $n \geq 2k$, $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$), and $t \geq 3$. Suppose the following conditions hold:*

1. $\mathrm{diam}(\Gamma_i^n) \geq t + 1$,
2. $n < \frac{(2t-1)(k-i)}{t-1}$,
3. *$P(v)$ holds for all $v \in \{3, \ldots, t\}$.*

Let $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$. Then for t even ($t = 2r$),

$$d(A, B) > 2r \Leftrightarrow (r-1)n - (2r-2)k + (2r-1)i < m < (2r+1)k - rn - 2ri,$$

and for t odd ($t = 2r + 1$),

$$d(A, B) > 2r + 1 \Leftrightarrow rn - 2rk + (2r+1)i < m < (2r+1)k - rn - 2ri.$$

Remark By Lemma 3.7, if (1) and (3) of Lemma 3.6 hold, then so does (2); hence, after Lemma 3.7 has been proved, we may omit (2) from the statement of Lemma 3.6. Similarly, after Lemma 3.5 has been proved, we may also omit (3).

Proof of Lemma 3.6 We proceed by induction on t. For the base case, suppose the conditions of the lemma hold for $t = 3$. Then by Corollary 3.4 and $P(3)$,

$$\begin{aligned} d(A, B) = 3 &\Leftrightarrow 0 \leq m \leq n - 2k + 3i, \\ d(A, B) = 2 &\Leftrightarrow m \geq 3k - n - 2i. \end{aligned}$$

Therefore the result for $t = 3$ follows. Now let $t = 2r > 3$ and suppose the lemma holds for $t - 1$. Suppose also that the conditions of the lemma hold for t. Then the conditions of the lemma also hold for $t - 1$, and so by the inductive assumption,

$$\begin{aligned} d(A, B) > 2r - 1 \Leftrightarrow (r - 1)n - (2r - 2)k + (2r - 1)i &< m \\ &< (2r - 1)k - (r - 1)n - (2r - 2)i. \end{aligned} \tag{1}$$

Now by $P(2r)$, we have that

$$\begin{aligned} d(A, B) = 2r \Leftrightarrow (2r + 1)k - rn - 2ri &\leq m \\ &\leq (2r - 1)k - (r - 1)n - (2r - 2)i - 1, \end{aligned} \tag{2}$$

and therefore we obtain the result for $t = 2r$ by combining (1) and (2). To complete the induction, it remains to show the result for $t + 1 = 2r + 1$ which can be done using similar arguments to the above and is left to the reader. □

Lemma 3.7 *Let* $n \geq 2k$, $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$ *(*$i \neq 0$*, if* $n = 2k$*), and* $t \geq 3$*. Suppose* $P(v)$ *holds for all* $v \in \{3, \ldots t\}$ *and* $\mathrm{diam}(\Gamma_i^n) \geq t + 1$*. Then*

$$n < \frac{(2t - 1)(k - i)}{t - 1}.$$

Proof We proceed by induction on t. For the base case, assume that $P(3)$ holds and $\mathrm{diam}(\Gamma_i^n) \geq 4$. Suppose $n \geq \frac{5(k-i)}{2}$. Then using Corollary 3.4 and $P(3)$ we see that $d(A, B) \leq 3$ for all $A, B \in V(\Gamma_i^n)$, which contradicts the fact that $\mathrm{diam}(\Gamma_i^n) \geq 4$. Hence $n < \frac{5(k-i)}{2}$. This completes the base case. Now let $t = 2r \geq 4$ and assume that the lemma holds for $t - 1$. Suppose $P(v)$ holds for all $v \in \{3, \ldots t\}$ and $\mathrm{diam}(\Gamma_i^n) \geq t + 1$. Then $P(v)$ holds for all $v \in \{3 \ldots, t - 1\}$ and $\mathrm{diam}(\Gamma_i^n) \geq t$; hence, by the inductive assumption, we have $n < \frac{(2t-3)(k-i)}{t-2}$. Now by Lemma 3.6, for $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$, we see that (1) holds. Suppose $n \geq \frac{(2t-1)(k-i)}{t-1}$. Then using $P(2r)$, we see that $d(A, B) \leq 2r$, for all $A, B \in V(\Gamma_i^n)$, which contradicts the fact that $\mathrm{diam}(\Gamma_i^n) \geq 2r + 1$. Hence $n < \frac{(2t-1)(k-i)}{t-1}$. To complete the induction, it remains to show the result for $t + 1 = 2r + 1$ which can be done using similar arguments to the above and is left to the reader. □

Lemma 3.8 *Suppose $n \geq 2k, i \in \{0, \dots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$), and* $\operatorname{diam}(\Gamma_i^n) \geq 3$. *Let $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$. If $n < \frac{5(k-i)}{2}$, then*

$$d(A, B) = 3 \Leftrightarrow 0 \leq m \leq n - 2k + 3i,$$

and if $n \geq \frac{5(k-i)}{2}$, then

$$d(A, B) = 3 \Leftrightarrow 0 \leq m \leq 3k - n - 2i - 1.$$

Proof Suppose $n < \frac{5(k-i)}{2}$. If $d(A, B) = 3$, then by Lemma 3.2(2), we have $m \leq n-2k+3i$. For the converse, assume $m \leq n-2k+3i$. Since $n < \frac{5(k-i)}{2}$, we have that $m < 3k-n-2i$, and so by Corollary 3.4, we have $d(A, B) > 2$. Choose a neighbour A_1 of A such that $|A_1 \cap B|$ is maximal. If $i < m$, then $|A_1 \cap B| = i+k-m$, and if $i > m$, then $|A_1 \cap B| = m+k-i$. In both cases (for $i < m$ use $m \leq n-2k+3i$, and for $i > m$ use $m \geq 0$), we have $|A_1 \cap B| \geq 3k-2i-n$. Hence by Lemma 3.3(1), we have $d(A_1, B) = 2$, and so $d(A, B) = 3$. Now suppose $n \geq \frac{5(k-i)}{2}$. If $d(A, B) = 3$, then by Corollary 3.4, we have $m < 3k-n-2i$. For the converse, assume $m < 3k-n-2i$. Then by Corollary 3.4, we have $d(A, B) > 2$, and using the fact that $n \geq \frac{5(k-i)}{2}$, we see that $m \leq n - 2k + 3i$. Now as above, we show that $d(A, B) = 3$ by constructing a path of length 3 between A and B. □

Corollary 3.9 *Let $n > 2k$. For $i = 0$,*

$$\operatorname{diam}(\Gamma_0^n) = 3 \Leftrightarrow \frac{5k-1}{2} \leq n < 3k - 1,$$

and for $i \in \{1, \dots, \lfloor k/2 \rfloor\}$,

$$\operatorname{diam}(\Gamma_i^n) = 3 \Leftrightarrow \frac{5(k-i)-1}{2} \leq n < 3k - 2i.$$

Proof Suppose $\operatorname{diam}(\Gamma_0^n) = 3$. By Theorem 1.3(1), we have $n < 3k - 1$ and by Corollary 3.4, for $A, B \in V(\Gamma_0^n)$ with $m := |A \cap B| \neq 0$,

$$d(A, B) > 2 \Leftrightarrow m < 3k - n. \tag{3}$$

Now $\operatorname{diam}(\Gamma_0^n) = 3$ implies that for all $A, B \in V(\Gamma_0^n)$ with $d(A, B) > 2$, we must have $d(A, B) = 3$. Thus using Lemma 3.8, we deduce that either $n \geq \frac{5k}{2}$ or $3k-n-1 = n-2k$. Conversely, suppose $\frac{5k-1}{2} \leq n < 3k-1$. Then by Theorem 1.3(1), we have $\operatorname{diam}(\Gamma_0^n) \geq 3$. Hence by Corollary 3.4, for $A, B \in V(\Gamma_0^n)$ with $m := |A \cap B| \neq 0$, we see that (3) holds. Now using Lemma 3.8 and the fact that $n \geq \frac{5k-1}{2}$, we see that $d(A, B) \leq 3$ for all $A, B \in V(\Gamma_0^n)$. Hence $\operatorname{diam}(\Gamma_0^n) = 3$. This proves the first statement of the corollary. The statement for $i > 0$ can be proved using similar arguments to the above and is left to the reader. □

Proof of Theorem 1.4(2b) See Corollary 3.9. □

We need one final lemma before we can prove Lemma 3.5.

Lemma 3.10 *Let $n \geq 2k$, $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$ ($i \neq 0$, if $n = 2k$), and $r \geq 2$. Let $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$.*

1. *Suppose $d(A, B) \geq 2r$, P(v) holds for $v \in \{3, \ldots, 2r-1\}$, and*
$$(2r+1)k - rn - 2ri \leq m < (2r-1)k - (r-1)n - (2r-2)i. \tag{4}$$
Then $d(A, B) = 2r$.
2. *Suppose $d(A, B) \geq 2r+1$, P(v) holds for $v \in \{3, \ldots, 2r\}$, and*
$$(r-1)n - (2r-2)k + (2r-1)i < m \leq rn - 2rk + (2r+1)i. \tag{5}$$
Then $d(A, B) = 2r+1$.

Proof Suppose the conditions in the first part of the lemma hold. Choose a neighbour A_1 of A such that $|A_1 \cap B|$ is minimal. If $k - m < i$, then using Lemma 3.3, we have $d(A, B) = 2$; therefore $k - m \geq i$, in which case $|A_1 \cap B| = 3k - i - n - m$. Now rearranging (4) gives

$$\begin{aligned}(2r-3)i + (r-2)n - (2r-4)k &< |A_1 \cap B| \\ &\leq (2r-1)i + (r-1)n - (2r-2)k.\end{aligned}$$

Therefore using Lemma 3.7 (with $t = 2r-1$) and $P(2r-1)$, we have $d(A_1, B) = 2r-1$, and hence $d(A, B) = 2r$. This proves the first part of the lemma. Now suppose the conditions in the second part of the lemma hold. Choose a neighbour A_1 of A such that $|A_1 \cap B|$ is maximal. Then (5) implies $m > i$, and so $|A_1 \cap B| = i + k - m$. Now rearranging (5) implies

$$(2r+1)k - rn - 2ri \leq |A_1 \cap B| < (2r-1)k - (2r-2)i - (r-1)n.$$

Therefore using Lemma 3.7 (with $t = 2r$) and $P(2r)$, we have $d(A_1, B) = 2r$, and hence $d(A, B) = 2r+1$. □

Proof of Lemma 3.5 For $v = 3$, the result is proved in Lemma 3.8. For $v \geq 4$, we proceed by induction on v. Here $d = \mathrm{diam}(\Gamma_i^n) \geq v$. Firstly, we prove the base case $v = 4$. Suppose $n < \frac{7(k-i)}{3}$. If $d(A, B) = 4$, then by Lemma 3.2(1), we have $m \geq 5k - 2n - 4i$. Since $d(A, B) > 2$, by Corollary 3.4, we also have $m < 3k - n - 2i$. For the converse, assume $5k - 2n - 4i \leq m < 3k - n - 2i$. Note that since $P(3)$ already holds, by the remark following Lemma 3.6, in order to use Lemma 3.6 (with $t = 3$), we only need to ensure that the first condition of the lemma is satisfied, which is by the assumption of Lemma 3.5. Now using Lemma 3.6 and the fact that $n < \frac{7(k-i)}{3}$, we see that $d(A, B) > 3$, and so by Lemma 3.10, we have $d(A, B) = 4$. Now suppose $n \geq \frac{7(k-i)}{3}$. If $d(A, B) = 4$, then using Lemma 3.6 (with $t = 3$), we have $m > n - 2k + 3i$. Also $d(A, B) > 2$, so by Corollary 3.4, we have $m < 3k - n - 2i$. For the converse, assume $n - 2k + 3i < m < 3k - n - 2i$. Then using Lemma 3.6, we see that $d(A, B) > 3$. Now using the fact that $n \geq \frac{7(k-i)}{3}$, we have $5k - 2n - 4i \leq m < 3k - n - 2i$, and so by Lemma 3.10, we have $d(A, B) = 4$. This completes the base case.

Now let $r \geq 2$ and assume that $P(v)$ holds for all $v \in \{3, \ldots, 2r\}$. We will show that $P(2r+1)$ holds. Suppose $n < \frac{(4r+1)(k-i)}{2r}$. If $d(A, B) = 2r+1$, then by Lemma 3.2(2), we have $m \leq rn - 2rk + (2r+1)i$. Since $d(A, B) > 2r - 1$, by Lemma 3.6 we also have $m > (r-1)n - (2r-2)k + (2r-1)i$. For the converse, assume that

$$(r-1)n - (2r-2)k + (2r-1)i < m \leq rn - 2rk + (2r+1)i. \tag{6}$$

Then using Lemma 3.6 and the fact that $n < \frac{(4r+1)(k-i)}{2r}$, we have $d(A, B) > 2r$, and so by Lemma 3.10, we have $d(A, B) = 2r + 1$. Now suppose that $n \geq \frac{(4r+1)(k-i)}{2r-1}$. If $d(A, B) = 2r + 1$, then by Lemma 3.6, we have $m < (2r+1)k - rn - 2ri$. Since $d(A, B) > 2r - 1$, by Lemma 3.6 we also have $m > (r-1)n - (2r-2)k + (2r-1)i$. For the converse, assume that

$$(r-1)n - (2r-2)k + (2r-1)i < m < (2r+1)k - rn - 2ri.$$

Then using Lemma 3.6, we have $d(A, B) > 2r$. Now using the fact that $n \geq \frac{(4r+1)(k-i)}{2r-1}$, we see that (6) holds, and so by Lemma 3.10, we have $d(A, B) = 2r+1$. This shows $P(2r+1)$. To complete the induction, it remains to show $P(2r+2)$ which can be done using similar arguments to the above and is left to the reader. □

Proof of Theorem 1.4(1) and (2c) By Theorem 1.3(1) and Corollary 3.9, we see that Theorem 1.4(1) holds when $\operatorname{diam}(\Gamma_0^n)$ is 2 or 3. For $t \geq 4$, an equivalent restatement of Theorem 1.4(1) and (2c) is that for $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$,

$$R(t): \quad \operatorname{diam}(\Gamma_i^n) = t \Leftrightarrow \frac{(2t-1)(k-i)-1}{t-1} \leq n < \frac{(2t-3)(k-i)-1}{t-2}. \tag{7}$$

We prove (7) by induction on t. The base case will be to show that

$$\operatorname{diam}(\Gamma_i^n) = 4 \Leftrightarrow \frac{7(k-i)-1}{3} \leq n < \frac{5(k-i)-1}{2}. \tag{8}$$

Suppose $\operatorname{diam}(\Gamma_i^n) = 4$. Then by Theorem 1.3(1) and Corollary 3.9, $n < \frac{5(k-i)-1}{2}$. Note that by the remark following Lemma 3.6, in order to use Lemma 3.6, we only need to ensure that the first condition of the lemma is satisfied. Now by Lemma 3.6, for $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$, we have

$$d(A, B) > 3 \Leftrightarrow n - 2k + 3i < m < 3k - n - 2i. \tag{9}$$

Now since $\operatorname{diam}(\Gamma_i^n) = 4$, $P(4)$ holds by Lemma 3.5 and using $P(4)$, we obtain that either $n \geq \frac{7(k-i)}{3}$ or $n - 2k + 3i + 1 = 5k - 2n - 4i$. Conversely, suppose that the right-hand side of (8) holds. Since $n < \frac{5(k-i)-1}{2}$, by Theorem 1.3(1) and Corollary 3.9, we have $\operatorname{diam}(\Gamma_i^n) \geq 4$, and by Lemma 3.6, we see that (9) holds. Now using (9), $P(4)$ and the fact that $n \geq \frac{7(k-i)-1}{3}$, we obtain that for all $A, B \in V(\Gamma_i^n)$ with $d(A, B) > 3$, we have $d(A, B) = 4$, and thus $\operatorname{diam}(\Gamma_i^n) = 4$. This completes the base case.

Now let $r \geq 3$ and suppose $R(t)$ holds for all $t \in \{4, \ldots, 2r-2\}$. We will show that $R(2r-1)$ holds. Suppose $\operatorname{diam}(\Gamma_i^n) = 2r-1$. Then by Theorem 1.3(1), Corollary 3.9, and the inductive assumption, we have $n < \frac{(4r-5)(k-i)-1}{2r-3}$. Now by Lemma 3.6, for $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$, we have

$$d(A, B) > 2r-2 \Leftrightarrow (r-2)n - (2r-4)k + (2r-3)i \\ < m < (2r-1)k - (r-1)n - (2r-2)i. \tag{10}$$

Now $\mathrm{diam}(\Gamma_i^n) = 2r-1$ implies that for all $A, B \in V(\Gamma_i^n)$ with $d(A, B) > 2r-2$, we must have $d(A, B) = 2r-1$. Hence using $P(2r-1)$, we obtain that either $(2r-2)n \geq (4r-3)(k-i)$, or

$$(2r-1)k - (r-1)n - (2r-2)i - 1 = (r-1)n - (2r-2)k + (2r-1)i.$$

Conversely, suppose that the right-hand side of (7) holds with $t = 2r-1$. Since $n < \frac{(4r-5)(k-i)-1}{2r-3}$, using Theorem 1.3(1), Corollary 3.9 and the inductive assumption, we have $\mathrm{diam}(\Gamma_i^n) \geq 2r-1$, and by Lemma 3.6, we see that (10) holds. Now using (10), $P(2r-1)$ and the fact that $n \geq \frac{(4r-3)(k-i)-1}{2r-2}$, we obtain that for all $A, B \in V(\Gamma_i^n)$ with $d(A, B) > 2r-2$, we have $d(A, B) = 2r-1$, and thus $\mathrm{diam}(\Gamma_i^n) = 2r-1$. This shows $R(2r-1)$. To complete the induction, it remains to show $R(2r)$ which can be proved similarly and is left to the reader. □

We now prove some lemmas required to prove Theorem 1.4(3).

Lemma 3.11 *Let $n \geq 2k$, $i \in \{\lfloor k/2 \rfloor + 1, \ldots, k-1\}$, and suppose $A, B \in V(\Gamma_i^n)$ with $|A \cap B| =: m > i$. Then $d(A, B) = 2$.*

Proof Since $m > i$, we can choose a k-subset C containing i elements from $A \cap B$, and since $i > k/2$ implies $|I \setminus (A \cup B)| = n - 2k + m > k - i$, we can also choose C to contain $k-i$ elements from $I \setminus (A \cup B)$. Then $|C| = k$ and $A - C - B$ is a path of length 2, and hence $d(A, B) = 2$. □

Now let $n \geq 2k$, $i \in \{\lfloor k/2 \rfloor + 1, \ldots, k-1\}$, and suppose that $A, B \in V(\Gamma_i^n)$ with $|A \cap B| =: m < i$. Let j_m be the least positive integer such that $j_m(k-i) + m \geq i$, i.e. $j_m = \lceil \frac{i-m}{k-i} \rceil$. Construct a path

$$A - A_1 - \cdots - A_{j_m}, \tag{11}$$

where we set $A_0 := A$, and for $l \in \{1, \ldots, j_m\}$, A_l is chosen so that $|A_{l-1} \cap A_l| = i$ and $|A_l \cap B|$ is maximal. Observe that $|A_l \cap B| = |A_{l-1} \cap B| + (k-i) = m + l(k-i)$, for any $l \in \{1, \ldots, j_m\}$.

Lemma 3.12 *Let $n \geq 2k$, $i \in \{\lfloor k/2 \rfloor + 1, \ldots, k-1\}$, and suppose $A, B \in V(\Gamma_i^n)$ with $|A \cap B| =: m < i$. Then $d(A, B) \leq j_m + 1$.*

Proof Consider the path in (11). If $j_m(k-i) + m = i$, then $|A_{j_m} \cap B| = i$. Hence $A - A_1 - \cdots - A_{j_m} - B$ is a path of length $j_m + 1$. If $|A_{j_m} \cap B| = j_m(k-i) + m > i$, then let $\alpha := j_m(k-i) + m - i > 0$. Since $i > k/2$, this implies

$$|A_{j_m - 1} \cup B| + \alpha = 3k - 2i \leq n,$$

and hence we can choose a vertex A'_{j_m} which is the vertex A_{j_m} but with α elements from $(A_{j_m} \cap B) \setminus (A_{j_m} \cap A_{j_m-1})$ replaced with α elements which are not in $A_{j_m-1} \cup B$. Then $A - A_1 - \cdots - A_{j_m-1} - A'_{j_m} - B$ is a path of length $j_m + 1$, and hence $d(A, B) \leq j_m + 1$. □

By Lemma 3.12 we know that if $A, B \in V(\Gamma_i^n)$ with $|A \cap B| := m < i$, then $d(A, B) \leq j_m + 1 = \lceil \frac{k-m}{k-i} \rceil$. We will now show that this inequality is in fact an equality.

Lemma 3.13 *Let $n \geq 2k$, $i \in \{\lfloor k/2 \rfloor + 1, \ldots, k-1\}$ and suppose $A, B \in V(\Gamma_i^n)$ with $|A \cap B| := m$. Then $d(A, B) \geq \lceil \frac{k-m}{k-i} \rceil$.*

Proof Suppose $d(A, B) = l + 1$ and $A - C_1 - \cdots - C_l - B$ is a shortest path. It can be seen using a short inductive argument that C_l can differ from A by at most $l(k-i)$ points. Using this and the fact that $|C_l \cap B| \leq |A \cap B| + |C_l \setminus A|$, we get $|C_l \cap B| \leq m + l(k-i)$. On the other hand, $|C_l \cap B| = i$ so $i \leq m + l(k-i)$. This implies

$$l + 1 \geq \frac{k-m}{k-i},$$

from which the claim follows. □

Remark

1. If $m > i$, then $0 < \frac{k-m}{k-i} < 1$, so Lemma 3.13 just states that $d(A, B) \geq 1$. However if $m < i$, then $\frac{k-m}{k-i} > 1$ and so Lemma 3.13 provides useful information.
2. Lemma 3.13 also holds for $i \leq \lfloor k/2 \rfloor$. For $m > i$, this is obvious by the first part of the remark. For $m < i$, this can be seen as follows. Since $m < i$, we have $d(A, B) \geq 2$. Also, $k - i \geq k/2$ which implies that

$$\frac{k-m}{k-i} \leq \frac{2(k-m)}{k} = 2\left(1 - \frac{m}{k}\right) \leq 2.$$

   Hence $d(A, B) \geq 2 \geq \frac{k-m}{k-i}$, from which the claim follows.

The following corollary is a consequence of Lemmas 3.12 and 3.13.

Corollary 3.14 *Let $n \geq 2k$, $i \in \{\lfloor k/2 \rfloor + 1, \ldots, k-1\}$, and suppose $A, B \in V(\Gamma_i^n)$ with $|A \cap B| := m < i$. Then $d(A, B) = \lceil \frac{k-m}{k-i} \rceil$.*

Proof of Theorem 1.4(3) Let $A, B \in V(\Gamma_i^n)$ with $|A \cap B| = m \geq 0$. If $m < i$, then by Corollary 3.14, $d(A, B) = \lceil \frac{k-m}{k-i} \rceil \geq 2$, and if $m > i$, then by Lemma 3.11, $d(A, B) = 2$. As m increases, the value of the function $\lceil \frac{k-m}{k-i} \rceil$ decreases. Hence the diameter of Γ_i^n is obtained when $m = 0$, which gives $\lceil \frac{k}{k-i} \rceil$. □

4 Action of S_n on (k, l)-partitions of I

We begin this section by finding the orbitals of the action of S_n on $I^{(k,l)}$. We then prove a result from which Theorem 1.5 is deduced and then use Theorem 1.5 to deduce Theorem 1.1(2). We continue by proving a sequence of results which lead up to the proof of Theorem 1.2(2), and we conclude the section by proving Theorem 1.3(2).

Proposition 4.1 *Suppose that $n = kl$ where $k > 1, l > 1$. Let $A, B, C, D \in I^{(k,l)}$. Then the pairs (A, B) and (C, D) lie in the same orbital of S_n if and only if there exist $l \times l$ permutation matrices P and Q such that $I_{CD} = P\, I_{AB}\, Q$.*

Proof Write $A = A_1|\ldots|A_l$, $B = B_1|\ldots|B_l$, $C = C_1|\ldots|C_l$, and $D = D_1|\ldots|D_l$. Suppose (A, B) and (C, D) lie in the same orbital of S_n. Then there exists $g \in S_n$ such that $A^g = C$ and $B^g = D$. Let $C' := C'_1|\ldots|C'_l$ be the partition C but with the components C_i permuted in such a way that $A_i^g = C'_i$. Similarly, let D' be the partition $D := D'_1|\ldots|D'_l$ but with the components D_i permuted in such a way that $B_i^g = D'_i$. Then there exists permutation matrices P^{-1} and Q^{-1} such that $I_{C'D'} = P^{-1} I_{CD} Q^{-1}$. Now for any $i, j \in \{1, \ldots, l\}$, we have that $|C'_i \cap D'_j| = |A_i \cap B_j|$, and hence $I_{CD} = P I_{C'D'} Q = P I_{AB} Q$. For the converse, suppose $I_{CD} = P I_{AB} Q$ for some $l \times l$ permutation matrices P and Q. Then after permuting the A_is and B_js suitably, we may suppose $I_{AB} = I_{CD}$. Now since S_n is n-transitive, we can pick $g \in S_n$ such that $(A_i \cap B_j)^g = C_i \cap D_j$ for all $i, j \in \{1, \ldots, l\}$ (note that $|\cup_{i,j\in\{1,\ldots,l\}} (A_i \cap B_j)| = n$). Then $(A, B)^g = (C, D)$, and so (A, B) and (C, D) lie in the same orbital. □

Definition 4.2 Let $n = kl$, where $k, l > 1$. We define the $l \times l$ matrix

$$M_{kl} := \left(\begin{array}{c|cc} kI_{l-2} & 0 & 0 \\ \hline 0 & 1 & k-1 \\ 0 & k-1 & 1 \end{array}\right).$$

Remark If $A, B \in V(\Gamma_{[M_{kl}]})$ are adjacent, then there exists a transposition in S_n sending A to B. Hence every path between A and B can be represented by a sequence of transpositions and the number of transpositions is the length of the path. For a path P between A and B, we will use σ_P to denote the product of the corresponding sequence of transpositions.

Proposition 4.3 *Let $n = kl$, where $k, l > 1$. Then* $\mathrm{diam}(\Gamma_{[M_{kl}]}) \geq \frac{l}{2} \left\lfloor \frac{k}{2} \right\rfloor$.

Proof Let A and B be two (k, l)-partitions such that

$$I_{AB} \sim \begin{pmatrix} \lfloor \frac{k}{2} \rfloor & \lceil \frac{k}{2} \rceil & 0 & \ldots & 0 & 0 \\ 0 & \lfloor \frac{k}{2} \rfloor & \lceil \frac{k}{2} \rceil & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & \lfloor \frac{k}{2} \rfloor & \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil & 0 & \ldots & \ldots & 0 & \lfloor \frac{k}{2} \rfloor \end{pmatrix}. \tag{12}$$

Write $A = A_1|\ldots|A_l$ and $B = B_1|\ldots|B_l$. If $d(A, B) = r$ and P is a path of length r between A and B, then σ_P sends A to B and for each $i \in \{1, \ldots, l\}$ there exists $j \in \{1, \ldots, l\}$ such that σ_P sends A_i to B_j. Now $|A_i \cap B_j| \leq \lceil \frac{k}{2} \rceil$ and hence σ_P must move at least $k - \lceil \frac{k}{2} \rceil = \lfloor \frac{k}{2} \rfloor$ elements from each A_i. Therefore σ_P must move at least $l \lfloor \frac{k}{2} \rfloor$ elements in total. Any permutation which moves x elements cannot be expressed as a product of fewer than $x/2$ transpositions. Hence we must have $r \geq \frac{l}{2} \lfloor \frac{k}{2} \rfloor$, from which the result follows. □

Proof of Theorem 1.5 This follows as a corollary of Proposition 4.3. □

Proof of Theorem 1.1(2) Suppose $\mathrm{diam_O}(G, \Omega) \leq c$. Then by Theorem 1.5, we must have

$$\frac{l}{2}\left\lfloor \frac{k}{2} \right\rfloor \leq c. \tag{13}$$

Now since $\lfloor \frac{k}{2} \rfloor \geq 1$, (13) implies $l \leq 2c$, and since $l \geq 2$ it also implies $k \leq 2c + 1$, and hence $n = kl \leq 2c(2c + 1)$. □

Next we prove Theorem 1.2(2). Suppose $G := S_n$ or A_n and the orbital diameter of $(G, I^{(k,l)})$ is bounded by 5. By Theorem 1.1(2), we must have $1 < l \leq 10$ and $1 < k \leq 11$, and we can make these values more precise using Proposition 4.3. In order to prove Theorem 1.2(2), for each possible pair (k, l), we must either determine that the diameters of all orbital graphs are bounded by 5 or find an orbital graph of diameter greater than 5. In order to reduce the possibilities for the pair (k, l) we prove the following results. In what follows, for a permutation σ, the support of σ is $\mathrm{supp}(\sigma) := \{x : x^\sigma \neq x\}$. For $l = 3$, we prove

Proposition 4.4 $\mathrm{diam}(\Gamma_{[M_{k3}]}) \geq 2 \lfloor \frac{k}{2} \rfloor$.

Proof Let $A := A_1|A_2|A_3$ and $B := B_1|B_2|B_3$ be two $(k, 3)$-partitions with

$$I_{AB} = \begin{pmatrix} \lfloor \frac{k}{2} \rfloor & 0 & \lceil \frac{k}{2} \rceil \\ \lceil \frac{k}{2} \rceil & \lfloor \frac{k}{2} \rfloor & 0 \\ 0 & \lceil \frac{k}{2} \rceil & \lfloor \frac{k}{2} \rfloor \end{pmatrix}.$$

Let $d(A, B) = r$ and suppose that P is a path of length r between A and B. As in the proof of Proposition 4.3, σ_P must move at least $\lfloor \frac{k}{2} \rfloor$ elements from each A_i. If there exists i such that σ_P moves all k elements from A_i, then σ_P moves at least $k + 2\lfloor \frac{k}{2} \rfloor$ elements in total, and hence $r \geq \lceil \frac{k}{2} + \lfloor \frac{k}{2} \rfloor \rceil = k \geq 2 \lfloor \frac{k}{2} \rfloor$. Now assume that σ_P fixes at least one element from each A_i. Set $X_1 := A_1 \cap B_1$, $Y_1 := A_2 \cap B_2$ and $Z_1 := A_3 \cap B_3$, and let X_2, Y_2 and Z_2 be their complements in A_1, A_2 and A_3, respectively. Then the following information can be read off I_{AB} ($\sqcup$ denotes disjoint union):

$$|X_1| = |Y_1| = |Z_1| = \left\lfloor \frac{k}{2} \right\rfloor, |X_2| = |Y_2| = |Z_2| = \left\lceil \frac{k}{2} \right\rceil,$$
$$A_1 = X_1 \sqcup X_2, A_2 = Y_1 \sqcup Y_2, A_3 = Z_1 \sqcup Z_2,$$
$$B_1 = X_1 \sqcup Y_2, B_2 = Y_1 \sqcup Z_2, B_3 = Z_1 \sqcup X_2.$$

Now either ${A_1}^{\sigma_P} = B_1$ or ${A_1}^{\sigma_P} = B_3$. Suppose ${A_1}^{\sigma_P} = B_1$. Then ${A_i}^{\sigma_P} = B_i$ for all i. Now if $\alpha \in X_2$ and $\beta := \alpha^{\sigma_P}$, then $\beta \in X_1 \sqcup Y_2$, so $\beta \neq \alpha$. Also, if $\beta \in Y_2$ then $\beta^{\sigma_P} \in Y_1 \sqcup Z_2$, so $\beta^{\sigma_P} \neq \alpha$, and if $\beta \in X_1$, then $\beta^{\sigma_P} \in X_1 \sqcup Y_2$, so again $\beta^{\sigma_P} \neq \alpha$. Hence α is contained in a cycle of σ_P, of length at least 3. Repeating similar arguments, we see that if $\alpha \in Y_2$ or $\alpha \in Z_2$, then α is contained in a cycle of σ_P, of length at least 3. By the remark following Definition 4.2, we have that r is greater then or equal to the minimal length of a factorisation of σ_P by transpositions. Using the fact that the minimal length of a factorisation of a permutation σ by transpositions is $|\mathrm{supp}(\sigma)| - c_\sigma$, where c_σ is the number of non-trivial cycles in the disjoint cycle decomposition of σ, we get $r \geq |\mathrm{supp}(\sigma_P)| - c_{\sigma_P}$. Suppose that there are s disjoint cycles of length at least 3 in the disjoint cycle decomposition of σ_P. Let $I_3 \subseteq \{1, \ldots, n\}$ be the union of the supports of these cycles. Then $|\mathrm{supp}(\sigma_P)| - \sigma_P \geq |I_3| - s$. Furthermore, $s \leq \frac{1}{3}|I_3|$ and hence we get $r \geq \frac{2}{3}|I_3|$. On the other hand, the preceding discussion implies $|I_3| \geq |X_2 \cup Y_2 \cup Z_2| = 3\left\lceil \frac{k}{2} \right\rceil$. Combining the two, we obtain $r \geq 2\left\lceil \frac{k}{2} \right\rceil \geq 2\left\lfloor \frac{k}{2} \right\rfloor$. Now suppose ${A_1}^{\sigma_P} = B_3$. Then ${A_2}^{\sigma_P} = B_1$ and ${A_3}^{\sigma_P} = B_2$. Repeating similar arguments to the above, we can see that every element of $X_1 \cup Y_1 \cup Z_1$ must be contained in a cycle of length at least 3, and so as above, we have that

$$r \geq \frac{2}{3} 3 \left\lfloor \frac{k}{2} \right\rfloor = 2 \left\lfloor \frac{k}{2} \right\rfloor.$$

Therefore $d(A, B) \geq 2\left\lfloor \frac{k}{2} \right\rfloor$, from which the result follows. □

For $k = 3$, we have the following result.

Proposition 4.5 *For $l \geq 4$, $\mathrm{diam}(\Gamma_{[M_{3l}]}) \geq \left\lceil \frac{l}{2} \right\rceil + 1$.*

Proof Let A and B be two $(3, l)$-partitions of I with

$$I_{AB} = \begin{pmatrix} 1 & 2 & 0 & \ldots & & 0 \\ 0 & 1 & 2 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 & 2 \\ 2 & 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

as in (12). By Proposition 4.3, $d(A, B) \geq \lceil \frac{l}{2} \rceil$. We will prove the claim by showing that $d(A, B) > \lceil \frac{l}{2} \rceil$. Assume by contradiction that P is a path of length $\lceil \frac{l}{2} \rceil$ between A and B. Write $\sigma_P = a_1 \ldots a_{\left\lceil \frac{l}{2} \right\rceil}$, where each a_i is a transposition, and write $A = A_1 | \ldots | A_l$. By the proof of Proposition 4.3, σ_P moves at least one element from each A_i. It follows that if l is even, then $|\mathrm{supp}(\sigma_P)| = l$ and if l is odd, then $|\mathrm{supp}(\sigma_P)| \in \{l, l+1\}$. In all cases, $\mathrm{supp}(\sigma_P) = \cup_{i=1}^{\left\lceil \frac{l}{2} \right\rceil} \mathrm{supp}(a_i)$, and now a careful analysis, whose details are left to the reader, shows that in all cases, there exists $1 \leq i \leq \left\lceil \frac{l}{2} \right\rceil$ and $1 \leq r \neq s \leq l$ such that $a_i = (x\, y)$ for some $x \in A_r$ and $y \in A_s$;

moreover, the supports of a_i and a_j are disjoint for all $1 \leq j \leq \left\lceil \frac{l}{2} \right\rceil$ such that $j \neq i$, and $|\mathrm{supp}(\sigma) \cap A_r| = |\mathrm{supp}(\sigma) \cap A_s| = 1$. In particular, a_i commutes with every other transposition and we can write $\sigma_P = a_i \sigma'_P$ where $\sigma'_P := a_1 \cdots a_{i-1} a_{i+1} \cdots a_{\left\lceil \frac{l}{2} \right\rceil}$. Thus σ_P sends A_r to $(A_r \backslash \{x\}) \cup \{y\}$ and A_s to $(A_s \backslash \{y\}) \cup \{x\}$. Therefore $I_{AB} \sim N$, where N is the $l \times l$ matrix $\begin{pmatrix} \mathrm{M} & 0 \\ 0 & \mathrm{X} \end{pmatrix}$, with $M := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and where X is an $(l-2) \times (l-2)$ matrix. Then there exist permutation matrices P and Q such that $P^{-1} I_{AB} = NQ$. Recall that multiplication by P^{-1} on the left permutes the rows of I_{AB}, and multiplication by Q on the right permutes the columns of N. Note that the entries of $P^{-1} I_{AB} = NQ$ take values from the set $\{0, 1, 2\}$. Firstly, suppose $(NQ)_{11} = 2$. Then the first and second rows of $P^{-1} I_{AB}$ must be $(2, 0, \ldots, 0, 1)$ and $(1, 2, 0, \ldots, 0)$, respectively. This implies that $(NQ)_{12} = 0$ and $(NQ)_{22} = 2$, which is not possible since N does not contain a column of the form $(0, 2, \ldots)^{\mathrm{T}}$. Now suppose that $(NQ)_{11} = 1$. Then the first and second rows of $P^{-1} I_{AB}$ must be $(1, 2, 0, \ldots, 0)$ and $(2, 0, \ldots, 0, 1)$, respectively. This implies that $(NQ)_{12} = 2$ and $(NQ)_{22} = 0$, which is again not possible. Finally suppose that $(NQ)_{11} = 0$. Then $(NQ)_{21} = 0$, and there exist $i \in \{2, \ldots, l\}$ such that $(NQ)_{1i} = 1$ and $(NQ)_{2i} = 2$. However this is a contradiction since the matrix I_{AB} has the property that if $(I_{AB})_{r1} = (I_{AB})_{s1} = 0$ and $(I_{AB})_{rt} = 1$, then $(I_{AB})_{st} = 0$, where $t \in \{2, \ldots, l\}$. Therefore we must have $d(A, B) \geq \left\lceil \frac{l}{2} \right\rceil + 1$. □

We also have a small improvement in the lower bound in Proposition 4.3 which is obtained as follows.

Proposition 4.6 *Let $A, B \in V(\Gamma_{[M_{kl}]})$ with I_{AB} as in (12). If P is a path between A and B and $\sigma_P \in S_{kl}$ moves exactly $l \left\lfloor \frac{k}{2} \right\rfloor$ points from A, then $d(A, B) \geq \left\lfloor \frac{k}{2} \right\rfloor (l-1)$, and hence* $\mathrm{diam}(\Gamma_{[M_{kl}]}) \geq \left\lfloor \frac{k}{2} \right\rfloor (l-1)$.

Proof Let $A := A_1 | \ldots | A_l$ and $B := B_1 | \ldots | B_l$. Since I_{AB} is given by (12), we can write $A_i = X_i \sqcup Y_i$ for all $1 \leq i \leq l$, where $|X_i| = \left\lfloor \frac{k}{2} \right\rfloor$. Then $B_i = X_{i+1} \sqcup Y_i$ for all $1 \leq i \leq l-1$, and $B_l = X_1 \sqcup Y_l$. By the proof of Proposition 4.3, σ_P moves at least $\left\lfloor \frac{k}{2} \right\rfloor$ elements from each A_i. Hence by the assumption, exactly $\left\lfloor \frac{k}{2} \right\rfloor$ elements are moved from each A_i. Therefore ${A_i}^{\sigma_P} = B_i$, for all i. In particular, σ_P sends X_i to X_{i+1}, for $i \in \{1, \ldots, l-1\}$, and X_l to X_1. Hence if we write $\sigma_P = C_1 \cdots C_r$, where the C_i are non-trivial disjoint cycles, then we can conclude that each C_i has length which is a multiple of l, say $t_i l$, and $r \leq \left\lfloor \frac{k}{2} \right\rfloor$. Then $\sum_{i=1}^{r} t_i l = \left\lfloor \frac{k}{2} \right\rfloor l$ and σ_P cannot be expressed as a product of fewer than

$$\sum_{i=1}^{r} (t_i l - 1) = \left\lfloor \frac{k}{2} \right\rfloor l - r$$

transpositions. The expression on the right-hand side is minimal when $r = \left\lfloor \frac{k}{2} \right\rfloor$ and hence $d(A, B) \geq \left\lfloor \frac{k}{2} \right\rfloor (l-1)$. □

Corollary 4.7 *$diam(\Gamma_{[M_{kl}]}) \geq min \left\{ \left\lceil \frac{l}{2} \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \right\rceil, \left\lfloor \frac{k}{2} \right\rfloor (l-1) \right\}$.*

Table 8 Possible pairs (k, l) with $\mathrm{diam}_{\mathrm{O}}(G, I^{(k,l)}) \leq 5$

l	2	3	4	5	6, 7, 8	9
k	≤ 11	≤ 6	≤ 5	≤ 3	≤ 3	2

Proof If $A, B \in V(\Gamma_{[M_{kl}]})$ and P is a path between A and B, then σ_P moves at least $l\left\lfloor \frac{k}{2} \right\rfloor$ points from A. Now if σ_P moves exactly $l\left\lfloor \frac{k}{2} \right\rfloor$ points, then by Proposition 4.6, $\mathrm{diam}(\Gamma_{[M_{kl}]}) \geq \left\lfloor \frac{k}{2} \right\rfloor (l-1)$. Otherwise, σ_P moves at least $l\left\lfloor \frac{k}{2} \right\rfloor + 1$ points and hence cannot be expressed as a product of fewer than $\frac{l\left\lfloor \frac{k}{2} \right\rfloor + 1}{2}$ transpositions. Therefore $\mathrm{diam}(\Gamma_{[M_{kl}]}) \geq \left\lceil \frac{l}{2}\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \right\rceil$. □

We conjecture that Proposition 4.4 can be generalised; more precisely that $\mathrm{diam}(\Gamma_{[M_{kl}]}) \geq \left\lfloor \frac{k}{2} \right\rfloor (l-1)$ for arbitrary values of k and l.

Using Propositions 4.4, 4.5, and Corollary 4.7, we obtain the following result.

Proposition 4.8 *Let $G := S_n$ or A_n and $\Omega := I^{(k,l)}$, where $n = kl$, $k > 1$, $l > 1$, and suppose that (k, l) is not given by one of the values in Table 8. Then* $\mathrm{diam}_{\mathrm{O}}(G, \Omega) \geq 6$.

For the case $l = 2$, we find the diameters of all orbital graphs of $(S_{k2}, I^{(k,2)})$ as stated in Theorem 1.6. Recall that for $l = 2$, the orbital graphs are $\Gamma_{[M_i]}$, where $i \in \{1, \ldots, \lfloor k/2 \rfloor\}$ and

$$M_i = \begin{pmatrix} i & k-i \\ k-i & i \end{pmatrix}.$$

Note that the matrix M_i is determined by $(M_i)_{11}$ and if $A, B \in V(\Gamma_{[M_i]})$, then the matrix I_{AB} is determined by $(I_{AB})_{11}$. In order to prove Theorem 1.6 we will need to prove the following criterion which relates distances in the graphs $\Gamma_{[M_i]}$ to distances in the graphs Γ_i^{2k} from Sect. 3. In what follows, for a k-subset X of $I := \{1, \ldots, 2k\}$, we will use the notation X^c to refer to the complement in I of the set X.

Lemma 4.9 *Let $i \in \{1, \ldots, \lfloor k/2 \rfloor\}$ and $A, B \in V(\Gamma_{[M_i]})$, where $A := A_1|A_2$ and $B := B_1|B_2$. Suppose $d_{\Gamma_{[M_i]}}(A, B) \geq r \geq 2$. Then $d_{\Gamma_{[M_i]}}(A, B) = r$ if and only if there exist $l, m \in \{1, 2\}$ such that $d_{\Gamma_i^{2k}}(A_l, B_m) = r$. In particular, $diam(\Gamma_{[M_i]}) \geq r$ if and only if $diam(\Gamma_i^{2k}) \geq r$.*

Proof Suppose $d_{\Gamma_{[M_i]}}(A, B) = r$ and let $A - C_1 - \cdots - C_{r-1} - B$ be a shortest path, where $C_i := X_i|Y_i$. Then there exist $\{l, m\} \in \{1, 2\}$, and $Z_i \in \{X_i, Y_i\}$ for $i \in \{1, \ldots, r-1\}$, such that $|A_l \cap Z_1| = i$, $|Z_i \cap Z_{i+1}| = i$ for $i \in \{1, \ldots, r-2\}$, and $|Z_{r-1} \cap B_m| = i$. Hence $A_l - Z_1 - \cdots - Z_{r-1} - B_m$ is a path of length r in Γ_i^{2k}. If $d_{\Gamma_i^{2k}}(A_l, B_m) =: t < r$ and $A_l - D_1 - \ldots D_{t-1} - B_m$ is a shortest path, then

$$A = A_l|A_l^c - D_1|D_1^c - \cdots - D_{t-1}|D_{t-1}^c - B_m|B_m^c = B$$

is a path of length t in $\Gamma_{[M_i]}$, which contradicts the fact that $d_{\Gamma_{[M_i]}}(A, B) < r$. Hence $d_{\Gamma_i^{2k}}(A_l, B_m) = r$. Conversely, suppose there exist $l, m \in \{1, 2\}$ such that

$d_{\Gamma_i^{2k}}(A_l, B_m) = r$. Let $A_l - X_1 - \cdots - X_{r-1} - B_m$ be a shortest path in Γ_i^{2k}. Then

$$A = A_l|A_l^c - X_1|X_1^c - \cdots - X_{r-1}|X_{r-1}^c - B_m|B_m^c = B$$

is a path of length r in $\Gamma_{[M_i]}$, and since $d_{\Gamma_{[M_i]}}(A, B) \geq r$, we have $d_{\Gamma_{[M_i]}}(A, B) = r$. □

The following lemma follows directly from Lemmas 3.3 and 3.5.

Lemma 4.10 *Suppose $n = 2k$. Let $i \in \{1, \dots, \lfloor k/2 \rfloor\}$ and $A, B \in V(\Gamma_i^n)$ with $m := |A \cap B| \neq i$. Suppose $d := \mathrm{diam}(\Gamma_i^n) \geq 2$. If $2i < k$, then*

$$d(A, B) = 2 \Leftrightarrow m \geq k - 2i,$$

and if $2i \geq k$, then $d(A, B) = 2$. Furthermore for $v \in \{3 \dots, d\}$, if $(2v-1)i < k$, then

$$\begin{aligned} d(A, B) = v &\Leftrightarrow k - vi \leq m < k - (v-2)i, && v \text{ even} \\ d(A, B) = v &\Leftrightarrow (v-2)i < m \leq vi, && v \text{ odd}, \end{aligned}$$

and if $(2v-1)i \geq k$, then

$$\begin{aligned} d(A, B) = v &\Leftrightarrow (v-1)i < m < k - (v-2)i, && v \text{ even} \\ d(A, B) = v &\Leftrightarrow (v-2)i < m < k - (v-1)i, && v \text{ odd}. \end{aligned}$$

Proof of Theorem 1.6(1) and (2a) Firstly, note that for $k < 4$, all orbital graphs have diameter 1. Suppose $k \geq 4$ and let $A, B \in V(\Gamma_{[M_i]})$ with $(I_{AB})_{11} =: j$, where $j \in \{1, \dots, \lfloor k/2 \rfloor\}$ and $j \neq i, k-i$. Then $d_{\Gamma_{[M_i]}}(A, B) \geq 2$ and by Lemma 4.9, $d_{\Gamma_{[M_i]}}(A, B) = 2$ if and only if there exist $l, m \in \{1, 2\}$ such that $d_{\Gamma_i^{2k}}(A_l, B_m) = 2$. Suppose $2i < k$. Then by Lemma 4.10,

$$d_{\Gamma_{[M_i]}}(A, B) = 2 \Leftrightarrow \text{ either } j \geq k - 2i \text{ or } j \leq 2i. \tag{14}$$

Therefore $\mathrm{diam}(\Gamma_{[M_i]}) = 2$ if and only if $2i \geq k - 2i - 1$. Now if $2i \geq k$, then by Lemma 4.10, we have $\mathrm{diam}(\Gamma_{[M_i]}) = 2$. We now obtain the result by gathering together the values of i for which the diameter is 2. □

Lemma 4.11 *Let $i \in \{1, \dots, \lfloor k/2 \rfloor\}$ and $A, B \in V(\Gamma_{[M_i]})$ with $(I_{AB})_{11} =: j$, where $j \in \{1, \dots, \lfloor k/2 \rfloor\}$ and $j \neq i, k-i$. Suppose $d_{\Gamma_{[M_i]}}(A, B) \geq 3$. If $5i < k$, then*

$$d_{\Gamma_{[M_i]}}(A, B) = 3 \Leftrightarrow \text{ either } 2i < j \leq 3i \text{ or } k - 3i \leq j < k - 2i, \tag{15}$$

and if $5i \geq k$, then

$$d_{\Gamma_{[M_i]}}(A, B) = 3 \Leftrightarrow 2i < j < k - 2i. \tag{16}$$

Proof Note that we must have $2i < k$ (else the diameter is 2), and hence (14) holds. Then the proof follows from Lemmas 4.9, 4.10, and (14). □

In order to prove Theorem 1.6(2b), we need the following.

Lemma 4.12 *Let* $t \geq 2$ *and suppose* $\operatorname{diam}(\Gamma_{[M_i]}) \geq t+1$. *If* $t = 2$, *then* $2i < k$ *and if* $t \geq 3$, *then* $(2t-1)i < k$.

Proof If $2i \geq k$, then by Lemma 4.10, we have $\operatorname{diam}(\Gamma_{[M_i]}) = 2$, hence the result for $t = 2$. If $t \geq 3$, then by Lemma 4.9, $\operatorname{diam}(\Gamma_i^{2k}) \geq t+1$ and the result follows from Lemma 3.7. □

Proposition 4.13 *Let* $i \in \{1, \ldots, \lfloor k/2 \rfloor\}$. *Then*

$$\operatorname{diam}(\Gamma_{[M_i]}) = 3 \Leftrightarrow \frac{k-1}{6} \leq i < \frac{k-1}{4}.$$

Proof Suppose $\operatorname{diam}(\Gamma_{[M_i]}) = 3$. Then $\operatorname{diam}(\Gamma_{[M_i]}) > 2$, so by Theorem 1.6(2a), we must have $i < \frac{k-1}{4}$. Now by Lemma 4.12, we see that (14) holds. Using (14) and Lemma 4.11, we see that $3i \geq k - 3i - 1$. Conversely if $\frac{k-1}{6} \leq i < \frac{k-1}{4}$, then by Theorem 1.6(2a), $\operatorname{diam}(\Gamma_{[M_i]}) \geq 3$. Now by Lemma 4.12, we see that (14) holds. Using (14) and Lemma 4.11, and the fact that $i \geq \frac{k-1}{6}$, we see that for all $A, B \in V(\Gamma_i^n)$ with $d(A, B) > 2$, we must have $d(A, B) = 3$, and thus $\operatorname{diam}(\Gamma_{[M_i]}) = 3$. □

Lemma 4.14 *Let* $r \geq 3$ *and* $A, B \in V(\Gamma_{[M_i]})$ *with* $(I_{AB})_{11} =: j$, *where* $j \in \{1, \ldots, k/2\}$ *and* $j \neq i, k-i$. *Suppose* $d_{\Gamma_{[M_i]}}(A, B) \geq r$.

1. *If* $(2r-1)i < k$, *then* $d_{\Gamma_{[M_i]}}(A, B) = r$ *if and only if either* $k - ri \leq j < k - (r-1)i$ *or* $(r-1)i < j \leq ri$.
2. *If* $(2r-1)i \geq k$, *then* $d_{\Gamma_{[M_i]}}(A, B) = r$ *if and only if* $(r-1)i < j < k - (r-1)i$.

Proof We proceed by induction or r. The base case $r = 3$ has been done in Lemma 4.11. Now suppose $d_{\Gamma_{[M_i]}}(A, B) \geq r \geq 4$ and that the lemma holds for all $s < r$. Then using Lemma 4.12 and the inductive hypothesis to determine the pairs of vertices for which the distance is strictly less than r, we obtain that

$$d_{\Gamma_{[M_i]}}(A, B) \geq r \Leftrightarrow (r-1)i < j < k - (r-1)i. \tag{17}$$

For j as in (17), by Lemmas 4.9 and 4.10, if $(2r-1)i < k$, then

$$d_{\Gamma_{[M_i]}}(A, B) = r \Leftrightarrow \text{ either } k - ri \leq j < k - (r-2)i \text{ or } (r-2)i < j \leq ri,$$

and if $(2r-1)i \geq k$, then

$$\begin{aligned} d_{\Gamma_{[M_i]}}(A, B) = r \Leftrightarrow \text{ either } &(r-1)i < j < k - (r-2)i \text{ or} \\ &(r-2)i < j < k - (r-1)i. \end{aligned}$$

We obtain the result by combing the inequalities above with (17). □

The next corollary is a direct consequence of the proof of Lemma 4.14.

Corollary 4.15 *Let* $r \geq 3$ *and* $A, B \in V(\Gamma_{[M_i]})$ *with* $(I_{AB})_{11} =: j$*, where* $j \in \{1, \ldots, k/2\}$ *and* $j \neq i, k-i$*. If* $\mathrm{diam}(\Gamma_{[M_i]}) \geq r$*, then*

$$d_{\Gamma_{[M_i]}}(A, B) \geq r \Leftrightarrow (r-1)i < j < k-(r-1)i.$$

We now prove Theorem 1.6(2b).

Proof of Theorem 1.6(2b) Suppose $\mathrm{diam}(\Gamma_{[M_i]}) \neq 2$. It suffices to show that for $r \geq 3$,

$$\mathrm{diam}(\Gamma_{[M_i]}) = r \Leftrightarrow \frac{k-1}{2r} \leq i < \frac{k-1}{2r-2}.$$

We proceed by induction on r. The base case $r = 3$ is done in Proposition 4.13. Now let $r \geq 4$ and suppose for the inductive hypothesis that

$$\mathrm{diam}(\Gamma_{[M_i]}) = s \Leftrightarrow \frac{k-1}{2s} \leq i < \frac{k-1}{2s-2}, \tag{18}$$

for all $s \in \{3, \ldots, r-1\}$. Suppose $\mathrm{diam}(\Gamma_{[M_i]}) = r$. Then $\mathrm{diam}(\Gamma_{[M_i]}) > r-1$, and so by Theorem 1.6(2a), Proposition 4.13 and (18), we must have $i < \frac{k-1}{2r-2}$. By Corollary 4.15, we have that for all $A, B \in V(\Gamma_{[M_i]})$ with $(I_{AB})_{11} =: j$, where $j \in \{1, \ldots, \lfloor k/2 \rfloor\}$ and $j \neq i, k-i$,

$$d_{\Gamma_{[M_i]}}(A, B) \geq r \Leftrightarrow (r-1)i < j < k-(r-1)i.$$

Now $\mathrm{diam}(\Gamma_{[M_i]}) = r$ implies that for all $A, B \in V(\Gamma_{[M_i]})$ with $d(A, B) \geq r$, we must have $d(A, B) = r$. Using this and Lemma 4.14, we see that either $(2r-1)i \geq k$, or $(2r-1)i < k$ and $ri \geq k - ri - 1$. In either case, $i \geq \frac{k-1}{2r}$. Conversely if $\frac{k-1}{2r} \leq i < \frac{k-1}{2r-2}$, then again by Theorem 1.6(2a), Proposition 4.13 and (18), we have $\mathrm{diam}(\Gamma_{[M_i]}) \geq r$. Then using Corollary 4.15, Lemma 4.14, and the fact that $i \geq \frac{k-1}{2r}$, we see that for all $A, B \in V(\Gamma_{[M_i]})$ with $d(A, B) \geq r$, we have $d(A, B) = r$, and thus $\mathrm{diam}(\Gamma_{[M_i]}) = r$. □

To complete the proof of Theorem 1.2(2), we need to establish which of the values in Table 8 in Proposition 4.8 actually yield $\mathrm{diam}_{\mathrm{O}}(G, I^{(k,l)}) \leq 5$. To do this, we have used MAGMA. We have devised two codes; the first takes as inputs the values k and l and returns the diameters of all orbital graphs of $(G, I^{(k,l)})$. In order to be able to deal with as high values of $n = kl$ as possible, the code is designed to calculate the collapsed adjacency matrix of each orbital graph as outlined in Sect. 2. As mentioned in Sect. 2, this code has limitations when the index gets large. In this case, we have devised a second code; fix two (k, l)-partitions A and B (usually with I_{AB} is as in (12)), and let $\sigma \in G := S_{kl}$ be a permutation sending A to B. Then any permutation sending A to B is of the form $w\sigma$, where $w \in G_A \cong S_k \wr S_l$. Let $\mathcal{R}$ be the set of permutations sending A to B. The code takes as inputs the values of k, l and σ and calculates for each $\tau \in \mathcal{R}$

the minimal number of transpositions $T(\tau)$ that τ can be a product of. The code returns the value $min_{\tau \in \mathcal{R}} T(\tau)$, which is (by the remark following Definition 4.2) the value of $d(A, B)$ in the orbital graph $\Gamma_{[M_{kl}]}$, and hence a lower bound for $\mathrm{diam}(\Gamma_{[M_{kl}]})$. Note that one can obtain the exact value of $\mathrm{diam}(\Gamma_{[M_{kl}]})$ by calculating $d(A, B)$ for all pairs (A, B), but for the results in this paper, only a lower bound is required.

Proof of Theorem 1.2(2) Table 8 lists the values of k and l which remain to be considered. For the case where $l = 2$, using Theorem 1.6 it can be seen that $\mathrm{diam}_{\mathrm{O}}(G, \Omega) \leq 5$ if and only if $2 \leq k \leq 11$. For $l \geq 3$, MAGMA has been used to verify the result as outlined in the paragraph after the proof of Theorem 1.6(2b). The data obtained can be found in Appendix 1. □

Although as $n = kl$ tends to infinity the diameters of the corresponding orbital graphs are unbounded, there do exist infinite families of orbital graphs with small diameters. We will now present one infinite family of diameter 2 graphs and prove Theorem 1.3(2). In order to do this, we will need the following preliminary definitions and lemma.

Definition 4.16 Let $A \in M_n(\mathbb{R}_{\geq 0})$ be an $n \times n$ matrix with non-negative real entries, and let $k \geq 1$ be an integer. We say that A is a *k-doubly stochastic* matrix if each row sum and each column sum is k. We say that A is *doubly stochastic* if A is 1-doubly stochastic.

Definition 4.17 Let $A \in M_n(\mathbb{R}_{\geq 0})$. A *diagonal* $D := (a_1, \ldots, a_n)$ of A is a sequence of n entries of A such that no two entries lie in the same row or column of A. A diagonal D is called *positive* if $a_i > 0$ for all i.

The following lemma can be found as Lemma 7.4.4 in [6].

Lemma 4.18 *Every doubly stochastic matrix has a positive diagonal.*

Proof of Theorem 1.3(2) Let $A, B \in V(\Gamma_{[M]})$ and write $A = A_1|\ldots|A_k$, $B = B_1|\ldots|B_k$. We will show that there exists a vertex C such that $I_{AC} \sim M \sim I_{CB}$. We will do this by describing an explicit construction for C. Define $I'_{AB} := \frac{1}{k} I_{AB}$. Then I'_{AB} is a doubly stochastic matrix. By Lemma 4.18, I'_{AB}, and therefore kI'_{AB}, has a positive diagonal. Moreover, the latter diagonal has positive integer entries, in particular at least 1. Denote this diagonal by $D = (a_{i_1 j_1}, \ldots, a_{i_k j_k})$, where for $r \in \{1, \ldots, k\}$, $a_{i_r j_r} := |S_{i_r j_r}| > 0$ and $S_{i_r j_r} := A_{i_r} \cap B_{j_r}$. Let C_1 be a k-subset which consists of exactly one element from each intersection $S_{i_r j_r}$, for $r \in \{1, \ldots, k\}$. Now $|C_1| = k$ and $|A_i \cap C_1| = 1 = |C_1 \cap B_i|$, for all $i \in \{1, \ldots, k\}$. Note that when we subtract 1 from each of the entries $a_{i_r j_r}$ in I_{AB} then we are left with a $(k-1)$-doubly stochastic matrix $I^{(1)}_{AB}$ where $(I^{(1)}_{AB})_{i_r j_r} = |(A_{i_r} \cap B_{j_r}) \setminus C_1|$, for $r \in \{1, \ldots, k\}$, and $(I^{(1)}_{AB})_{xy} = (I_{AB})_{xy}$, otherwise. As before we can use Lemma 4.18 to deduce that $I^{(1)}_{AB}$ has a positive integer diagonal, say $D' = (b_{t_1 u_1}, \ldots, b_{t_k u_k})$, where for $r \in \{1, \ldots, k\}$, either $b_{t_r u_r} = |(A_{t_r} \cap B_{u_r}) \setminus C_1|$ or $b_{t_r u_r} = |A_{t_r} \cap B_{u_r}|$. In the former case, define $T_{t_r u_r} := (A_{t_r} \cap B_{u_r}) \setminus C_1$, and in the latter case, define $T_{t_r u_r} := A_{t_r} \cap B_{u_r}$. Note that in either case $T_{t_r u_r} \cap C_1 = \emptyset$. We let C_2 be the set which consists of exactly one element from each of the sets $T_{t_r u_r}$, for $r \in \{1, \dots, k\}$. Then $|C_2| = k$, $|A_i \cap C_2| = 1 = |C_2 \cap B_i|$ for all $i \in \{1, \dots, k\}$, and the matrix $I^{(2)}_{AB}$ is $(k-2)$-doubly stochastic. We may repeat the above process to construct C_j at which stage we are left with a $(k-j)$-doubly stochastic matrix. Then we can construct C_{j+1} as long as $k - j \geq 1$, i.e. $j + 1 \leq k$. So we may construct sets $C_1, \dots, C_k$ such that $|C_j| = k$ and $|A_i \cap C_j| = 1 = |C_j \cap B_i|$ for all $i, j \in \{1, \dots, k\}$. The sets C_j are disjoint by construction. Now define $C := C_1|\dots|C_k$. Then $I_{AC} \sim M \sim I_{CB}$, and hence $A - C - B$ is a path of length 2. □

5 Action of G on $(G : H)$, H primitive on I

In this section, we prove Theorems 1.1(3), 1.2(3) and 1.3(3). The analysis of the actions of $G := S_n$ or A_n on $\Omega := (G : H)$, where H is primitive on $I = \{1, \dots, n\}$, rely on the following bound.

Lemma 5.1 *Let (G, Ω) be a finite primitive permutation group and $H := G_\alpha$. Let $\Delta := (\alpha, \beta)^G$ and $d := diam(\Gamma_{\Delta \cup \Delta^*})$. Then*

$$|\Omega| \leq 1 + 2|Y| + 2|Y|(2|Y| - 1) + \cdots + (2|Y|)(2|Y| - 1)^{d-1},$$

where $Y := \beta^H$. In particular (since $|Y| \leq |H|$), we have that

$$|\Omega| \leq 1 + 2|H| + 2|H|(2|H| - 1) + \cdots + (2|H|)(2|H| - 1)^{d-1}.$$

Proof Let v denote the valency of $\Gamma_{\Delta \cup \Delta^*}$. If Δ is self-paired, then $v = |\beta^H|$, and if Δ is not self-paired, then $v = 2|\beta^H|$. Now the result follows from the following inequality:

$$|\Omega| \leq 1 + v + v(v - 1) + v(v - 1)^2 + \cdots + v(v - 1)^{d-1}.$$

□

We will also need the following result (Corollary 1.2 in [9]) concerning the orders of primitive subgroups of S_n and A_n.

Lemma 5.2 *If H is a primitive subgroup of S_n not containing A_n, then $|H| < 3^n$. Moreover, if $n > 24$, then $|H| < 2^n$.*

Proof of Theorem 1.1(3) By Lemma 5.1

$$|G : H| \leq 1 + 2|H| + 2|H|(2|H| - 1) + \cdots + 2|H|(2|H| - 1)^{c-1} < (2|H|)^{c+1}. \tag{19}$$

Then using Lemma 5.2 and (19), either $n \leq 24$ or $n!/2 < 2^{(c+1)(n+1)+n}$. Now by (8.8) in [4, p.161], we have that $n!/2 > (n/e)^n$. If $n > 24$, then using this and the fact that $(n + 1)/n \leq \epsilon := 1.04$, we obtain

$$n < e\, 2^{\epsilon(c+1)+1} < 2^{\epsilon(c+1)+3}. \tag{20}$$

Note that (20) also holds for $n \leq 24$ (using $c \geq 1$) hence holds for all n. □

Next we prove Theorem 1.2(3). By Lemma 5.1, if (G, Ω) has an orbital graph of diameter d, then

$$|G| \leq |H| \left(1 + 2|H| \frac{(2|H| - 1)^d - 1}{2|H| - 2}\right). \tag{21}$$

So either $n \leq 24$ or, using Lemma 5.2,

$$|G| \leq 2^n \left(1 + 2^{n+1} \frac{(2^{n+1} - 1)^d - 1}{2^{n+1} - 2}\right). \tag{22}$$

Using (22) with $d = 5$, we see that $n \leq 173$ for $G = S_n$ and $n \leq 174$ for $G = A_n$. We can then use in-built functions in MAGMA to run through values of $n \leq 174$ and for each such n, iterate through the maximal subgroups of G and output only the primitive subgroups H which satisfy (21) with $d = 5$. This way we limit the number of possible pairs (n, H) for which $(G, (G : H))$ has an orbital graph of diameter $d \leq 5$; to 22 and 30 for $G = S_n$ and $G = A_n$, respectively. Now if $\mathrm{diam}_O(G, \Omega) \leq 5$, then there must exist an orbital graph of diameter d where $d \leq 5$, and hence we also obtain all possibilities of $(G, (G : H))$ for which the orbital diameter is bounded by 5. In order to rule out some of these possibilities, we prove the following result.

Proposition 5.3 *Let (G, Ω) be a finite primitive permutation group where $G := S_n$ or A_n and $\Omega := (G : H)$, where H is primitive on $I = \{1, \ldots, n\}$. Suppose there exists a subgroup $X \leq H$ such that $N_G(X) > N_H(X)$. If $N_G(X) \setminus N_H(X)$ contains an element of order 2, then set $a := 1$, otherwise set $a := 2$. There exists an orbital graph of diameter d, where*

$$d > \frac{log(|G : H|)}{log(a|H|) - log(|X|)} - 1.$$

Proof Pick $g \in N_G(X) \setminus N_H(X)$. Let $\alpha := H$, $\beta := Hg$, and $k := |\beta^{G_\alpha}|$. Consider the orbital $\Delta := (\alpha, \beta)^G$ and let $d := diam(\Gamma_{\Delta \cup \Delta^*})$. If g has order 2, then g swaps α and β, and hence Δ is self-paired. By the proof of Lemma 5.1, we have that

$$|G : H| \leq 1 + ak + ak(ak - 1) + \cdots + ak(ak - 1)^{d-1} < (ak)^{d+1},$$

where a is as in the statement of the lemma. Rearranging for d, this implies

$$d > \frac{log(|G : H|)}{log(ak)} - 1. \tag{23}$$

Now $k = |G_\alpha : G_\alpha \cap g^{-1} G_\alpha g|$ and $|G_\alpha \cap g^{-1} G_\alpha g| \geq |X|$. Hence $k \leq \frac{|H|}{|X|}$. Using this in (23), we obtain the result. □

Table 9 Possible pairs (n, H) with $\mathrm{diam}_{\mathrm{O}}(S_n, \Omega) \le 5$, where $\Omega := (S_n : H)$ and H is primitive

n	H
5	$AGL_1(5)$
6	$PGL_2(5)$
7	$AGL_1(7)$
8	$PGL_2(7)$
9	$AGL_2(3)$
10	$P\Gamma L_2(9)$
11	$AGL_1(11)$
12	$PGL_2(11)$
14	$PGL_2(13)$
18	$PGL_2(17)$

Table 10 Possible pairs (n, H) with $\mathrm{diam}_{\mathrm{O}}(A_n, \Omega) \le 5$, where $\Omega := (A_n : H)$ and H is primitive

n	H
5	D_{10}
6	$PSL_2(5)$
7	$PSL_2(7)$
8	$2^3 : PSL_2(7)$
9	$3^2 : 2A_4$, $P\Gamma L_2(8)$
10	M_{10}
11	M_{11}
12	M_{12}
13	$PSL_3(3)$
14	$PSL_2(13)$
15	$PSL_4(2)$
16	$2^4 : PSL_4(2)$
24	M_{24}

In light of Proposition 5.3, picking a large subgroup $X \le H$ such that $N_G(X) > N_H(X)$ can determine the existence of an orbital graph of large diameter. We have used MAGMA to devise a code which, given G and H, finds the largest such X and calculates the lower bounds in Proposition 5.3. This code has been used with the possibilities of $(G, (G : H))$ obtained using (21) with $d \le 5$, to obtain the following result.

Proposition 5.4 *Let (G, Ω) be a finite primitive permutation group where $G := S_n$ or A_n and $\Omega := (G : H)$, where H is primitive on $I = \{1, \ldots, n\}$ and $H \ne A_n$. If n and H are in not in Tables 9 or 10, then* $\mathrm{diam}_{\mathrm{O}}(G, \Omega) \ge 6$.

To deal with the remaining cases in Tables 9 and 10, we have again used MAGMA to devise a code which takes as input a value of n and outputs each possible primitive group H from Tables 9 and 10, the diameters of the corresponding orbital graphs, and

Table 11 Possible pairs (n, H) such that (S_n, Ω) has an orbital graph of diameter 2, where $\Omega := (S_n : H)$ and H is primitive

n	H
5	$AGL_1(5)$
6	$PGL_2(5)$
7	$AGL_1(7)$
8	$PGL_2(7)$
9	$AGL_2(3)$
10	$P\Gamma L_2(9)$
12	$PGL_2(11)$

Table 12 Possible pairs (n, H) such that (A_n, Ω) has an orbital graph of diameter 2, where $\Omega := (A_n : H)$ and H is primitive

n	H
5	D_{10}
6	$PSL_2(5)$
7	$PSL_3(2)$
8	$ASL_3(2)$
9	$3^2 : 2A_4$, $P\Gamma L_2(8)$
10	M_{10}
11	M_{11}
12	M_{12}
13	$PSL_3(3)$
15	$PSL_4(2)$
16	$2^4 : PSL_4(2)$
24	M_{24}

their valencies. Again the method uses the collapsed adjacency matrix of each orbital graph to calculate the diameter, as outlined in Sect. 2.

Proof of Theorem 1.2(3) By Proposition 5.4, the only possible pairs (n, H) for which $\text{diam}_O(G, \Omega) \leq 5$ are those in Tables 9 and 10. Finally, MAGMA has been used to verify (by implementing method CAM as in Sect. 2) that out of the possible cases, the only cases for which $\text{diam}_O(G, \Omega) \leq 5$ are the ones listed in the theorem. The data obtained is given in Appendix 1. □

Proof of Theorem 1.3(3) Using (22) with $d = 2$, we obtain $n \leq 24$ for both $G = S_n$ and $G = A_n$. We then use in-built functions in MAGMA to run through values of $n \leq 24$ and for each such n, iterate through the maximal subgroups of G to obtain only the primitive ones H which satisfy (21) with $d = 2$. We conclude that if there exists an orbital graph of diameter 2, then n and H must be in Tables 11 or 12. Finally, MAGMA has been used to verify (by implementing method CAM as in Sect. 2) that the only pairs (n, H) for which diameter 2 graphs occur are the ones stated in the theorem. □

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Appendix 1: Bounds on orbital diameters

Table 13 contains results (obtained using MAGMA) which are used to verify that if $G := S_{kl}$ or A_{kl}, then $\mathrm{diam}(G, I^{(k,l)}) \leq 5$ only for values of k and l listed in Theorem 1.2(2). The table specifies the values of k and l, the rank of the action and the value of $\mathrm{diam}_{\mathrm{O}}(G, I^{(k,l)})$.

Tables 14 and 15 contain results (obtained using MAGMA) which are used to verify that if $G := S_n$ or A_n and $\Omega := (G : H)$, where H is primitive on $I = \{1, \ldots, n\}$

Table 13 Orbital diameters for $(G, I^{(k,l)})$

l	k	Rank	$\mathrm{diam}_{\mathrm{O}}(G, I^{(k,l)})$
3	2	3	2
	3	5	3
	4	9	4
	5	13	5
	6	22	6
4	2	5	3
	3	12	4
	4	43	6
	5	–	≥ 6
5	2	7	4
	3	31	6
	4	–	≥ 8
6	2	11	5
	3	–	≥ 6
7	2	15	6
	3	–	≥ 6
8	2	22	7
	3	–	≥ 7
9	2	30	8
10	2	–	≥ 9

Table 14 Orbital diameters for (S_n, Ω), where $\Omega := (S_n : H)$ and H is primitive

n	H	Rank	$\text{diam}_O(S_n, \Omega)$
5	$AGL_1(5)$	2	1
6	$PGL_2(5)$	2	1
7	$AGL_1(7)$	7	3
8	$PGL_2(7)$	5	3
9	$AGL_2(3)$	9	5
10	$P\Gamma L_2(9)$	10	4
11	$AGL_1(11)$	3347	≥ 8
12	$PGL_2(11)$	396	6

Table 15 Orbital diameters for (A_n, Ω), where $\Omega := (A_n : H)$ and H is primitive

n	H	Rank	$\text{diam}_O(A_n, \Omega)$
5	D_{10}	2	1
6	$PSL_2(5)$	2	1
7	$PSL_2(7)$	2	1
8	$2^3 : PSL_2(7)$	2	1
9	$3^2 : 2A_4$	12	5
9	$P\Gamma L_2(8)$	3	2
10	M_{10}	12	4
11	M_{11}	5	2
12	M_{12}	4	2
13	$PSL_3(3)$	126	6
15	$PSL_4(2)$	1687	≥ 6
16	$2^4 : PSL_4(2)$	151	6

$(H \neq A_n)$, then $\text{diam}(G, \Omega) \leq 5$ only for the pairs (n, H) listed in Theorem 1.2(3). The tables specify the pairs (n, H), the rank of the action, and the value of $\text{diam}_O(G, \Omega)$.

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